ERGODIC OPTIMIZATION FOR BETA-TRANSFORMATIONS

ZELAI HAO, YINYING HUANG, OLIVER JENKINSON, AND ZHIQIANG LI

Abstract. Ergodic optimization for beta-transformations $T_\beta(x) = \beta x \pmod 1$ is developed. If $\beta > 1$ is a beta-number, or such that the orbit-closure of 1 is not minimal, we show that the Typically Periodic Optimization Conjecture holds, establishing that there exists an open dense set of Hölder continuous functions such that for each function in this set, there exists a unique maximizing measure, this measure is supported on a periodic orbit, and the periodic locking property holds. It follows that typical periodic optimization is typical among the class of beta-transformations: it holds for a set of parameters $\beta > 1$ that is residual, and has full Lebesgue measure. A novelty of the approach is that a perturbation argument in parameter space is used, and a locally Hölder version of the Mañé lemma is established.

Contents

1. Introduction

Dynamical properties of *beta-transformations*, self-maps of the unit interval of the form $T_\beta(x) = \beta x \pmod 1$, for $\beta > 1$, have been studied since the foundational papers of Rényi [Re57] and Parry [Pa60]. There are close connections with aspects of number theory, in view of the link between T_β-orbits and *beta-expansions* of the form $\varepsilon_1/\beta + \varepsilon_2/\beta^2 + \varepsilon_3/\beta^3 + \cdots$ (see e.g. [AB07, Be86, CK04, DK02, DK03, FS92, Ka15, Sc80, Si03]), with symbolic dynamics, where the *beta-shift* (see e.g. [AJ09, Bl89, IT74, Sc97, Si76]) serves as a shift-invariant model for the dynamics of T_β, and with ergodic theory (see e.g. [Ho78, Sm73, Wa78]).

In this paper we initiate a study of *ergodic optimization* (see e.g. [Boc18, Je06, Je19] for overviews of this area) in the context of beta-transformations. In other words, for a given $\beta > 1$, and a given (bounded, Borel measurable) real-valued function ψ, we seek to understand those x that maximize the time average $\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi\bigl(T_\beta^i(x)\bigr)$, and those T_β-invariant (Borel

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probability) measures μ that maximize the space average $\int \psi \, \mathrm{d}\mu$. Since its beginnings in the 1990s, a motivating theme in ergodic optimization has been the prospect that, for sufficiently regular functions ψ, the maximizing orbit (or measure) is *typically periodic* (see e.g. [Bou00, CLT01, HO96a, HO96b, Je96, Je00, YH99]). A result of this kind had been conjectured by Yuan and Hunt [YH99, Conjecture 1.1] for dynamical systems that are Axiom A or uniformly expanding, and for functions ψ that are Lipschitz or C^1 smooth. This conjecture generated a sustained period of work over the following years, and was eventually proved by Contreras [Co16], who showed that if the dynamical system is an open (distance-)expanding map, then there is an open and dense set of Lipschitz functions whose maximizing measure is unique and periodic. Analogous results have subsequently been established in related settings (see [HLMXZ19, LZ25]), and more broadly the Typically Periodic Optimization (TPO) Conjecture (see [Je19]) envisages that this can be further extended to other dynamical systems[1].

Although beta-transformations share some features of open expanding maps, the setting has a number of technical challenges: most notably T_β is neither continuous nor open, its space of invariant probability measures need not be weak* compact, and the corresponding beta-shift is in general not of finite type. Nevertheless, in this note we are able to develop a satisfactory theory of ergodic optimization for beta-transformations (and hence, the elements of a more general theory of ergodic optimization for non-continuous dynamical systems), and in particular make progress towards proving a TPO Conjecture for beta-transformations. To describe this in more detail, let us fix some terminology and notation.

For a general Borel measurable map $T\colon X \to X$ on a compact metric space X, let $\mathcal{M}(X,T)$ denote the set of T-invariant Borel probability measures on X, and define the *maximum ergodic average* of a bounded Borel measurable function $\psi\colon X \to \mathbb{R}$ to be

$$Q(T,\psi) \coloneqq \sup_{\mu \in \mathcal{M}(X,T)} \int \psi \, \mathrm{d}\mu. \tag{1.1}$$

Any measure $\mu \in \mathcal{M}(X,T)$ that attains the supremum in (1.1) is called a (T,ψ)-*maximizing measure*, or simply a ψ-*maximizing measure*, and the set of such measures is denoted by

$$\mathcal{M}_{\max}(T,\,\psi) \coloneqq \biggl\{ \mu \in \mathcal{M}(X,T) : \int \psi \, \mathrm{d}\mu = Q(T,\psi) \biggr\}. \tag{1.2}$$

The orbit of a point $x \in X$ is called a *maximizing orbit* for (T,ψ) if

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi\bigl(T^i(x)\bigr) = Q(T,\psi).$$

Setting $I \coloneqq [0,1]$, for each $\beta > 1$ the *beta-transformation* $T_\beta\colon I \to I$ is defined by

$$T_\beta(x) \coloneqq \beta x - \lfloor \beta x \rfloor, \quad x \in I,$$

where $\lfloor \beta x \rfloor \coloneqq \max\{n \in \mathbb{Z} : n \leqslant \beta x\}$. In order to develop ergodic optimization in the setting of beta-transformations, an analysis of the set $\mathcal{M}(I,T_\beta)$ of T_β-invariant measures, going somewhat beyond the existing literature, is first required. For this, it is convenient to consider the *upper beta-transformation*[2] $U_\beta\colon I \to I$, defined by $U_\beta(0) \coloneqq 0$ and

$$U_\beta(x) \coloneqq \beta x - \lfloor \beta x \rfloor', \quad x \in (0,1],$$

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[1] A parallel programme, inspired by [HO96a, HO96b], envisages that, for suitable dynamical systems and function spaces, periodic optimization is typical in a *probabilistic sense*; for progress towards this, see [BZ16, DLZ24].

[2] Also known as the *left-continuous beta-transformation*, cf. [KS12, Definition 2.4].

where $\lfloor \beta x \rfloor' := \max\{n \in \mathbb{Z} : n < \beta x\}$. Both T_β and U_β have a countable infinity of periodic orbits, and these orbits coincide, except that the orbit of 1 under U_β (referred to as the *critical orbit*) is periodic for certain values of β (so-called *simple beta-numbers*) while for such β the T_β-orbit of 1 is pre-fixed, eventually landing on the point 0. Each periodic orbit supports a unique invariant probability measure, and such periodic measures are weak* dense in both $\mathcal{M}(I, T_\beta)$ and $\mathcal{M}(I, U_\beta)$ (see [Si76] and Section 4). The set $\mathcal{M}(I, U_\beta)$ is weak* compact, and equal to $\mathcal{M}(I, T_\beta)$ provided β is not a simple beta-number; if β is a simple beta-number then $\mathcal{M}(I, T_\beta)$ is not weak* compact, but is dense in $\mathcal{M}(I, U_\beta)$.[3]

In order to describe our results towards the TPO Conjecture for beta-transformations, let $C(I)$ denote the set of continuous real-valued functions on I, define

$$\mathscr{P}(\beta) \subseteq C(I)$$

to be the set of those $\phi \in C(I)$ with a (U_β, ϕ)-maximizing measure supported on a periodic orbit of U_β, and for $\alpha \in (0, 1]$ define

$$\mathscr{P}^\alpha(\beta) := \mathscr{P}(\beta) \cap C^{0,\alpha}(I),$$

where $C^{0,\alpha}(I)$ denotes the set of α-Hölder functions on I, equipped with its usual Banach norm (see Section 2). If $\phi \in \mathscr{P}^\alpha(\beta)$ satisfies $\operatorname{card} \mathcal{M}_{\max}(U_\beta, \phi) = 1$ and $\mathcal{M}_{\max}(U_\beta, \phi) = \mathcal{M}_{\max}(U_\beta, \psi)$ for all $\psi \in C^{0,\alpha}(I)$ sufficiently close to ϕ in $C^{0,\alpha}(I)$, we say that ϕ has the *(periodic) locking property*[4] in $C^{0,\alpha}(I)$, and define the corresponding *(periodic) locking set* to be the open subset $\mathrm{Lock}^\alpha(\beta) \subseteq C^{0,\alpha}(I)$ given by

$$\mathrm{Lock}^\alpha(\beta) := \big\{\phi \in \mathscr{P}^\alpha(\beta) : \phi \text{ satisfies the locking property in } C^{0,\alpha}(I)\big\}.$$

Following Parry [Pa60], the value $\beta > 1$ is said to be a *beta-number* if the critical orbit is preperiodic (i.e., either periodic or strictly preperiodic). We are able to prove that, for beta-numbers, the following version of the TPO Conjecture does hold:

Theorem 1.1. *For a beta-number $\beta > 1$ and any $\alpha \in (0, 1]$, the set $\mathrm{Lock}^\alpha(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

To make progress on the TPO Conjecture beyond the class of beta-numbers, it is useful to divide the values $\beta > 1$ into two categories, by defining β to be *emergent*[5] if the closure of the U_β-orbit of 1 is minimal (i.e., it contains no proper closed subset E satisfying $U_\beta(E) = E$). We can then prove the TPO Conjecture for all non-emergent values of β:

Theorem 1.2. *If $\beta > 1$ is non-emergent and $\alpha \in (0, 1]$, then $\mathrm{Lock}^\alpha(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

In fact it can be shown that the set of emergent numbers is small, being both topologically meagre and of zero Lebesgue measure, so we deduce that typical periodic optimization holds for typical values of β, in the following sense.

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[3] If β is not a simple beta-number then ergodic optimization, and in particular the TPO problem, is identical for T_β and U_β. For simple beta-numbers, the lack of compactness of $\mathcal{M}(I, T_\beta)$, and consequent absence of a maximizing measure for certain continuous functions (indeed such functions constitute a non-empty open subset of $C^{0,\alpha}(I)$, cf. Remark A.3) motivates the formulation of ergodic optimization in terms of U_β.

[4] The terminology follows [Boc19, BZ15], and is somewhat inspired by [Bou00, Je00].

[5] See Section 6 for further details about, and alternative characterisations of, the set of emergent numbers: the terminology reflects the fact that, for an emergent $\beta > 1$, the symbolic dynamics corresponding to the critical orbit is essentially different from that witnessed in beta-shifts with parameter strictly smaller than β, so is considered to have newly *emerged* at this particular β (cf. Remark 6.2).

Corollary 1.3. *Fix $\alpha \in (0,1]$. For a set of values $\beta > 1$ which is both residual and of full Lebesgue measure, the periodic locking set $\mathrm{Lock}^{\alpha}(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Note that Theorem 1.2 is complementary to Theorem 1.1, rather than strictly stronger, since *simple* beta-numbers (where 1 is periodic under U_β) are emergent. More generally, the emergent values of β are less amenable to analysis, and in order to state a result concerning them we first introduce the set

$$\mathrm{Crit}^{\alpha}(\beta) := \bigl\{\phi \in C^{0,\alpha}(I) : \text{ the orbit of 1 is a maximizing orbit for } (U_\beta, \phi)\bigr\},$$

reflecting the fact that the difficulty concerns the critical orbit.

For emergent β, it turns out that the space of α-Hölder functions can be covered by $\mathrm{Crit}^{\alpha}(\beta)$ and the closure of $\mathrm{Lock}^{\alpha}(\beta)$:

Theorem 1.4. *If $\beta > 1$ is emergent and $\alpha \in (0,1]$, then $C^{0,\alpha}(I)$ is equal to the union of $\mathrm{Crit}^{\alpha}(\beta)$ and the closure of the open set $\mathrm{Lock}^{\alpha}(\beta)$.*

A key feature of the proofs of the above theorems is a perturbation argument in the space of parameters $\beta > 1$, exploiting the fact that the dynamics for a given β is approximable by sub-systems corresponding to simple beta-numbers in $(1, \beta)$. A means of facilitating this argument, and a result of independent interest, is a *Mañé lemma* for beta-transformations (see Theorem 5.9, and cf. Theorem 5.10). Results of this kind have been a hallmark of ergodic optimization since [Bou00, CLT01], and assert that cohomology classes of suitably regular functions contain versions (so-called *revealed versions*, cf. [Je19]) for which the maximizing measures are readily apparent; in favourable settings the revealed version inherits the modulus of continuity of the original function (see e.g. [Bou00, Bou01, BJ02, CLT01, LZ25]). Our Mañé lemma is for Hölder continuous functions, and asserts the existence of *two* revealed versions, both of which enjoy one-sided continuity (one version is left-continuous, the other is right-continuous): the critical orbit introduces discontinuities, but away from this orbit both versions are locally Hölder. The method for proving the Mañé lemma is to some extent familiar, via the fixed point of a nonlinear operator analogous to that introduced by Bousch [Bou00], though the proof of the existence of this fixed point requires new techniques involving pointwise limits. The fixed point is a Borel measurable function that in general may not be left-continuous or right-continuous, but has one-sided limits everywhere, and is locally Hölder away from the critical orbit: this fixed point can then be *regularised*, yielding both a left-continuous and a right-continuous *sub-action*, allowing the definition of left-continuous and right-continuous revealed versions. In particular this allows us to establish Theorem 5.10, asserting that any maximizing measure must be supported within the set of maxima of these revealed versions.

The other main ingredient for proving Theorems 1.1, 1.2, and 1.4, is a set $\mathcal{R}^{\alpha}(\beta)$ of α-Hölder functions enjoying *good restrictions* (in terms of membership of periodic locking sets, see Definition 7.7) to various Cantor subsets on which T_β acts as an open expanding map: by exploiting Contreras' TPO theorem [Co16] for such maps, we are able to prove (see Proposition 7.8) that $\mathcal{R}^{\alpha}(\beta)$ is dense in $C^{0,\alpha}(I)$. This, together with Theorem 5.10, and a finer analysis of $\mathrm{Crit}^{\alpha}(\beta)$ in the case of emergent β, yields a proof of Theorems 1.1, 1.2, and 1.4. Note in particular that, by contrast with [Co16, HLMXZ19, LZ25], the method for proving our TPO theorems does not make explicit use of shadowing (indeed the shadowing property does not hold for beta-transformations, cf. [BGS25]).

Organisation of the paper. Section 2 collects together some notation used throughout the article. In Section 3 we establish various preliminary results about beta-transformations, beta-expansions, and the closely related beta-shifts, being particularly careful to distinguish the commonalities and differences between these systems. In so doing, we take the opportunity to clarify and correct some aspects of the published literature on beta-shifts. In Section 4, we prove the existence of (U_β, ψ)-maximizing measures for all values $\beta > 1$, and all continuous functions ψ; we also introduce the notion of limit-maximizing measure, and establish the equivalence between maximizing measures for the upper beta-transformation and the beta-shift, and limit-maximizing measures for the beta-transformation. In Section 5 we establish a Mañé lemma for beta-transformations, and develop a revelation theorem as its consequence. In Section 6 the notion of *emergent numbers* is introduced, and a number of characterisations are proved. In Section 7, we establish the main theorems on typical periodic optimization. In Appendix A, we prove a useful auxiliary result, that $\operatorname{Lock}^\alpha(\beta)$ is an open and dense subset of $\mathscr{P}^\alpha(\beta)$.

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2. NOTATION

We follow the convention that $\mathbb{N} \coloneqq \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$. For $x \in \mathbb{R}$, we define the floor function $\lfloor x \rfloor$ as the largest integer $\leqslant x$, and the strict floor function $\lfloor x \rfloor'$ as the largest integer $< x$. The cardinality of a set A is denoted by $\operatorname{card} A$.

The collection of all maps from a set X to a set Y is denoted by Y^X. The constant zero function $\mathbb{0}\colon X \to \mathbb{R}$ maps each $x \in X$ to 0.

Let (X, d) be a metric space. For subsets $A, B \subseteq X$, we set $d(A, B) \coloneqq \inf\{d(x, y) : x \in A,\, y \in B\}$, and $d(A, x) = d(x, A) \coloneqq d(A, \{x\})$ for each $x \in X$. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}(Y) \coloneqq \sup\{d(x, y) : x,\, y \in Y\}$, for each $\epsilon > 0$, denote the ϵ-neighbourhood of Y (in X) by $B(Y, \epsilon) \coloneqq \{x \in X : d(x, Y) < \epsilon\}$, and the closure of $B(Y, \epsilon)$ by $\overline{B}(Y, \epsilon)$. For each $y \in X$ and each $\epsilon > 0$, write $B(y, \epsilon) \coloneqq B(\{y\}, \epsilon)$.

Let $C(X)$ denote the space of continuous functions from X to $\mathbb{R}$, and $\mathcal{P}(X)$ the set of Borel probability measures on X. For $\phi\colon X \to \mathbb{R}$, we write $\|\phi\|_{\infty, X} \coloneqq \sup\{|\phi(x)| : x \in X\}$.

Let (X, d) be a compact metric space and $\alpha \in (0, 1]$. A function $\phi\colon X \to \mathbb{R}$ is called α-Hölder if

$$|\phi|_{\alpha, X} \coloneqq \sup\{|\phi(x) - \phi(y)|/d(x, y)^\alpha : x,\, y \in X,\, x \neq y\} < +\infty.$$

Denote by $C^{0,\alpha}(X)$ the set of real-valued α-Hölder functions ϕ on X, equipped with the Hölder norm $\|\cdot\|_{\alpha, X}$ given by

$$\|\phi\|_{\alpha, X} \coloneqq |\phi|_{\alpha, X} + \|\phi\|_{\infty, X},$$

which makes $C^{0,\alpha}(X)$ a Banach space. We will omit the subscript X whenever $X = [0, 1]$.

For a Borel measurable map $T\colon X \to X$, let $\mathcal{M}(X, T)$ denote the set of T-invariant Borel probability measures on X.

For a map $T\colon X \to X$ and a real-valued function $\phi\colon X \to \mathbb{R}$, define

$$S_n^T \phi(x) \coloneqq \sum_{i=0}^{n-1} \phi\bigl(T^i(x)\bigr) \quad \text{for } x \in X, n \in \mathbb{N}.$$

In the particular case where T is some beta-transformation T_β, we will write $S_n\phi = S_n^{T_\beta}\phi$ whenever there is no possibility of confusion. Note that by definition $S_0^T\phi \equiv 0$.

We write $I \coloneqq [0,1]$. In this paper we equip every subset of I with the usual Euclidean metric, denoted by d.

For a non-empty set $\mathcal{A}$, the sequence space $\mathcal{A}^{\mathbb{N}} = \{\{a_n\}_{n=1}^{+\infty} : a_n \in \mathcal{A} \text{ for all } n \in \mathbb{N}\}$ will be equipped with the product topology. For $t > 1$, the metric d_t on $\mathcal{A}^{\mathbb{N}}$ defined by $d_t(\{a_n\}_{n=1}^{+\infty}, \{b_n\}_{n=1}^{+\infty}) \coloneqq t^{-p}$, where p is the smallest positive integer with $a_p \neq b_p$, and $d_t(\{a_n\}_{n=1}^{+\infty}, \{b_n\}_{n=1}^{+\infty}) \coloneqq 0$ if $a_n = b_n$ for all $n \in \mathbb{N}$, generates the product topology on $\mathcal{A}^{\mathbb{N}}$.

Infinite sequences will be written as $A = a_1a_2\dots = \{a_n\}_{n\in\mathbb{N}}$, and finite sequences as $B = b_1b_2\dots b_k = \{b_n\}_{n=1}^k$. Denote $(b_1b_2\dots b_k)^\infty \coloneqq b_1b_2\dots b_kb_1b_2\dots b_kb_1b_2\dots$ and write $(b_1b_2\dots b_k)^m$ for the first km terms of $(b_1b_2\dots b_k)^\infty$, for $m \in \mathbb{N}$.

If $\mathcal{A} \subseteq \mathbb{R}$ is equipped with the order induced by $\mathbb{R}$, and $A, B \in \mathcal{A}^{\mathbb{N}}$, write $A \prec B$ when A has strictly smaller lexicographic order than B, i.e., $a_i = b_i$ for $1 \leqslant i \leqslant n-1$, and $a_n < b_n$, for some $n \in \mathbb{N}$. Write $A \preceq B$ to mean $A \prec B$ or $A = B$.

Define the (left) shift map

$$\sigma\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$$

by $\sigma(A) \coloneqq \{a_{n+1}\}_{n\in\mathbb{N}}$ for all $A = \{a_n\}_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

Let X be a topological space. For a map $T\colon X \to X$ and $x \in X$, denote the orbit of x by

$$\mathcal{O}^T(x) \coloneqq \{T^n(x) : n \in \mathbb{N}_0\}.$$

In this paper, we write $\mathcal{O}_\beta(x) \coloneqq \mathcal{O}^{T_\beta}(x)$ and $\mathcal{O}'_\beta(x) \coloneqq \mathcal{O}^{U_\beta}(x)$ if it does not cause confusion. If there exists $n \in \mathbb{N}$ with $T^n(x) = x$, then $\mathcal{O}(x)$ is called a periodic orbit and x a periodic point. We denote the set of periodic points of T by $\operatorname{Per}(T)$. There is a unique T-invariant Borel probability measure $\mu_{\mathcal{O}}$ supported on a periodic orbit $\mathcal{O}$, given by

$$\mu_{\mathcal{O}} \coloneqq \frac{1}{\operatorname{card}\mathcal{O}} \sum_{x\in\mathcal{O}} \delta_x.$$

For a topological space X, a point $a \in X$, and a function $f\colon X \to \mathbb{R}$, we write $\lim_{y\to a} f(y) = x^+$ if $\lim_{y\to a} f(y) = x$ and there exists a neighbourhood U of a such that $f|_{U\smallsetminus\{a\}} \geqslant x$. We define $\lim_{y\to a} f(y) = x^-$ similarly. We say that a sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ converges to a real number x^+ (written as $\lim_{n\to+\infty} x_n = x^+$) if $x_n \geqslant x$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = x$. Moreover, for a real number a, and a function $g\colon \mathbb{R} \to \mathbb{R}$, we denote $\lim_{y\searrow a} g(y) \coloneqq \lim_{y\to a} g|_{(a,+\infty)}(y)$, that is, the right-hand limit of g at a. We denote the left-hand limit $\lim_{y\nearrow a} g(y) \coloneqq \lim_{y\to a} g|_{(-\infty,a)}(y)$ similarly. More generally, if we replace $\mathbb{R}$ by a topological well-ordered set Y, we give the definitions and notations above similarly.

Fix a constant $\alpha \in (0,1]$, a compact metric space X, and a measurable map $T\colon X \to X$. We define subsets

$$\mathscr{P}^\alpha(T) \qquad \text{and} \qquad \operatorname{Lock}^\alpha(T)$$

of the set $C^{0,\alpha}(X)$ as follows: $\mathscr{P}^\alpha(T)$ is the set of those $\phi \in C^{0,\alpha}(X)$ with a ϕ-maximizing measure supported on a periodic orbit of T. If a function $\phi \in \mathscr{P}^\alpha(T)$ satisfies $\operatorname{card}\mathcal{M}_{\max}(T,\,\phi) = 1$ and $\mathcal{M}_{\max}(T,\,\phi) = \mathcal{M}_{\max}(T,\,\psi)$ for all $\psi \in C^{0,\alpha}(X)$ sufficiently close to ϕ in $C^{0,\alpha}(X)$, we say that ϕ has the *(periodic) locking property* in $C^{0,\alpha}(X)$. The set $\operatorname{Lock}^\alpha(T)$ is defined to consist of all $\phi \in \mathscr{P}^\alpha(T)$ satisfying the (periodic) locking property in $C^{0,\alpha}(X)$.

3. BETA-TRANSFORMATIONS

3.1. **Beta-transformations, beta-expansions, and beta-shifts.** Here we recall the definitions and basic properties of beta-transformations, beta-expansions, and beta-shifts, with a particular focus (see Definition 3.10) on a classification according to the behaviour of the orbit of the point 1.

Definition 3.1 (Beta-transformations). Given a real number $\beta > 1$, the *beta-transformation* $T_\beta \colon I \to I$ is defined by

$$T_\beta(x) \coloneqq \beta x - \lfloor \beta x \rfloor, \quad x \in I. \tag{3.1}$$

Recall that $\lfloor x \rfloor' = \max\{n \in \mathbb{Z} : n < x\}$ for $x \in \mathbb{R}$. The *upper beta-transformation* $U_\beta \colon I \to I$ is defined by $U_\beta(0) \coloneqq 0$ and

$$U_\beta(x) \coloneqq \beta x - \lfloor \beta x \rfloor', \quad x \in I \smallsetminus \{0\}. \tag{3.2}$$

Note that Kalle and Steiner [KS12, Definition 2.4] refer to the upper beta-transformation as the left-continuous beta-transformation.

Definition 3.2 (Beta-expansions). Given a real number $\beta > 1$, write

$$\mathcal{B} \coloneqq \{0,\, 1,\, \dots,\, \lfloor \beta \rfloor\},$$

and define the *β-expansion* of $x \in I$ to be the sequence

$$\underline{\varepsilon}(x,\beta) = \{\varepsilon_n(x,\beta)\}_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$$

given by

$$\varepsilon_n(x,\beta) \coloneqq \left\lfloor \beta T_\beta^{n-1}(x) \right\rfloor \quad \text{for all } n \in \mathbb{N}, \tag{3.3}$$

and define the *upper β-expansion* of $x \in I$ to be the sequence

$$\underline{\varepsilon}^*(x,\beta) = \{\varepsilon_n^*(x,\beta)\}_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$$

given by

$$\varepsilon_n^*(x,\beta) \coloneqq \left\lfloor \beta U_\beta^{n-1}(x) \right\rfloor' \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Remark 3.3. For a given $\beta > 1$, the beta-transformation and upper beta-transformation are related by

$$U_\beta(x) = \limsup_{y \to x} T_\beta(y).$$

The set D_β of points of discontinuity for T_β is

$$D_\beta \coloneqq T_\beta^{-1}(0) \smallsetminus \{0\} = U_\beta^{-1}(1) = \{j/\beta : j \in \mathbb{Z}\} \cap (0,1], \tag{3.5}$$

and this is precisely the set of points at which T_β and U_β differ, with $T_\beta(x) = 0$ and $U_\beta(x) = 1$ for all $x \in D_\beta$. Note that Blanchard [Bl89, p. 136] refers to the upper β-expansion $\underline{\varepsilon}^*(x,\beta)$ as a kind of *incorrect* β-expansion.

Lemma 3.4. *If $\beta > 1$, $n \in \mathbb{N}$, $a \in [0,1)$, and $b \in (0,1]$, then the following statements are true:*

- (i) $\lim_{x \searrow a} T_\beta^n(x) = T_\beta^n(a)^+$ *and* $\lim_{x \nearrow b} U_\beta^n(x) = U_\beta^n(b)^-$.
- (ii) $\varepsilon_n(\cdot,\beta)$ *is right-continuous on* $[0,1)$ *and* $\varepsilon_n^*(\cdot,\beta)$ *is left-continuous on* $(0,1]$.
- (iii) $T_\beta^n(0) = \varepsilon_n(0,\beta) = U_\beta^n(0) = \varepsilon_n^*(0,\beta) = 0$.
- (iv) $\lim_{x \nearrow b} T_\beta^n(x) = U_\beta^n(b)^-$ *and* $\lim_{x \nearrow b} \varepsilon_n(x,\beta) = \varepsilon_n^*(b,\beta)$.

Proof. Define functions $f, g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f(u) \coloneqq \beta u, \quad g_1(u) \coloneqq u - \lfloor u \rfloor, \quad g_2(u) \coloneqq u - \lfloor u \rfloor'.$$

Then f is continuous and strictly increasing, and for each $u \in \mathbb{R}$, we have

$$\lim_{x \nearrow u} g_1(x) = g_2(u)^-, \quad \lim_{x \searrow u} g_1(x) = g_1(u)^+, \quad \lim_{x \nearrow u} g_2(x) = g_2(u)^-. \tag{3.6}$$

(i) By (3.1), we have $T_\beta = g_1 \circ f$ and $T_\beta^n = g_1 \circ f \circ T_\beta^{n-1}$. By (3.6), $\lim_{x \searrow a} T_\beta(x) = T_\beta(a)^+$. Assume that $\lim_{x \searrow a} T_\beta^n(x) = T_\beta^n(a)^+$ when $n = k$. When $n = k+1$,

$$\lim_{x \searrow a} T_\beta^{k+1}(x) = \lim_{x \searrow a} T_\beta\bigl(T_\beta^k(x)\bigr) = T_\beta^{k+1}(a)^+.$$

By induction, for all $n \in \mathbb{N}$, $\lim_{x \searrow a} T_\beta^n(x) = T_\beta^n(a)^+$. We can prove $\lim_{x \nearrow b} U_\beta^n(x) = U_\beta^n(b)^-$ similarly.

(ii) By (3.3) and (3.4), we have $\varepsilon_n(\cdot, \beta) = \lfloor \cdot \rfloor \circ f \circ T_\beta^{n-1}$ and $\varepsilon_n^*(\cdot, \beta) = \lfloor \cdot \rfloor' \circ f \circ U_\beta^{n-1}$. Hence (ii) follows from (i) and the fact that $\lfloor \cdot \rfloor$ is right-continuous and $\lfloor \cdot \rfloor'$ is left-continuous.

(iii) follows immediately from (3.1), (3.3), (3.4), and $U_\beta(0) = 0$.

(iv) Since $T_\beta = g_1 \circ f$ and $U_\beta = g_2 \circ f$ (see (3.1) and (3.2)), by (3.6),

$$\lim_{x \nearrow b} T_\beta(x) = \lim_{x \nearrow b} (g_1 \circ f)(x) = \lim_{u \nearrow f(b)} g_1(u) = (g_2 \circ f)(b)^- = U_\beta(b)^-.$$

Assume that $\lim_{x \nearrow b} T_\beta^n(x) = U_\beta^n(b)^-$ holds for $n = k$. When $n = k+1$,

$$\lim_{x \nearrow b} T_\beta^{k+1}(x) = \lim_{x \nearrow b} T_\beta\bigl(T_\beta^k(x)\bigr) = \lim_{u \nearrow U_\beta^k(b)} T_\beta(u) = U_\beta\bigl(U_\beta^k(b)\bigr)^- = U_\beta^{k+1}(b)^-.$$

Hence the first part of (iv) follows by induction.

By (3.3) and the first part of (iv),

$$\lim_{x \nearrow b} \varepsilon_n(x, \beta) = \lim_{x \nearrow b} (\lfloor \cdot \rfloor \circ f)\bigl(T_\beta^{n-1}(x)\bigr) = \lim_{u \nearrow U_\beta^{n-1}(b)} (\lfloor \cdot \rfloor \circ f)(u) = \lim_{v \nearrow f(U_\beta^{n-1}(b))} \lfloor v \rfloor.$$

By the fact that $\lim_{x \nearrow u} \lfloor x \rfloor = \lfloor u \rfloor'$ for all $u \in \mathbb{R}$ and (3.4),

$$\lim_{v \nearrow f(U_\beta^{n-1}(b))} \lfloor v \rfloor = \bigl\lfloor \beta U_\beta^{n-1}(b)) \bigr\rfloor' = \varepsilon_n^*(b, \beta).$$

The second part of (iv) follows from the above two equalities. ☐

Lemma 3.5. *If $x \in I$, $\beta > 1$, and $n \in \mathbb{N}$, then the following statements are true:*

(i) $\lim_{\gamma \searrow \beta} T_\gamma^n(x) = T_\beta^n(x)^+$ *and* $\lim_{\gamma \nearrow \beta} U_\beta^n(x) = U_\beta^n(x)^-$.

(ii) $\varepsilon_n(x, \cdot)$ *is right-continuous and* $\varepsilon_n^*(x, \cdot)$ *is left-continuous.*

(iii) $\lim_{\gamma \nearrow \beta} T_\gamma^n(x) = U_\beta^n(x)^-$ *and* $\lim_{\gamma \nearrow \beta} \varepsilon_n(x, \gamma) = \varepsilon_n^*(x, \beta)$.

Proof. Without loss of generality we can assume that $x \neq 0$ (see Lemma 3.4 (iii)). Define functions $f, g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f(u) \coloneqq xu, \quad g_1(u) \coloneqq u - \lfloor u \rfloor, \quad g_2(u) \coloneqq u - \lfloor u \rfloor'.$$

Note that f is continuous and strictly increasing and

$$\lim_{x \nearrow u} g_1(x) = g_2(u)^-, \quad \lim_{x \searrow u} g_1(x) = g_1(u)^+, \quad \lim_{x \nearrow u} g_2(x) = g_2(u)^-. \tag{3.7}$$

(i) By (3.1), we have $T_\beta(x) = (g_1 \circ f)(\beta)$ and $T_\beta^n(x) = g_1\bigl(\beta T_\beta^{n-1}(x)\bigr)$. By (3.7), $\lim_{\gamma \searrow \beta} T_\gamma(x) = T_\beta(x)^+$. Since $\beta > 1$, if $\lim_{\gamma \searrow \beta} T_\gamma^{k-1}(x) = T_\beta^{k-1}(x)^+$ for some $k \in \mathbb{N}$, we have $\lim_{\gamma \searrow \beta} \gamma T_\gamma^{k-1}(x) =$

$\beta T_\beta^{k-1}(x)^+$. Then by (3.7), $\lim_{\gamma\searrow\beta} T_\gamma^k(x) = T_\beta^k(x)^+$. By induction, for each $\beta > 1$, $\lim_{\gamma\searrow\beta} T_\gamma^n(x) = T_\beta^n(x)^+$. Similarly, we have $\lim_{\gamma\nearrow\beta} U_\beta^n(x) = U_\beta^n(x)^-$.

(ii) By (i) and $\beta > 1$, we have $\lim_{\gamma\searrow\beta} \gamma T_\gamma^{n-1}(x) = \beta T_\beta^{n-1}(x)^+$ and $\lim_{\gamma\nearrow\beta} \gamma U_\gamma^{n-1}(x) = U_\beta^{n-1}(x)^-$. Since $\lfloor\cdot\rfloor$ is right-continuous and $\lfloor\cdot\rfloor'$ is left-continuous, by (3.3) and (3.4), (ii) follows.

(iii) Since $T_\beta(x) = (g_1 \circ f)(\beta)$ and $U_\beta(x) = (g_2 \circ f)(\beta)$, by (3.7),

$$\lim_{\gamma\nearrow\beta} T_\gamma(x) = \lim_{\gamma\nearrow\beta} g_1(\gamma x) = \lim_{u\nearrow f(\beta)} g_1(u) = g_2(f(\beta))^- = U_\beta(a)^-.$$

Assume that $\lim_{\gamma\nearrow\beta} T_\gamma^n(x) = U_\beta^n(x)^-$ holds for $n = k$. When $n = k+1$, by (3.7),

$$\lim_{\gamma\nearrow\beta} T_\gamma^{k+1}(x) = \lim_{\gamma\nearrow\beta} g_1\bigl(\gamma T_\gamma^k(x)\bigr) = \lim_{u\nearrow\beta U_\beta^k(x)} g_1(u) = g_2\bigl(\beta U_\beta^k(x)\bigr)^- = U_\beta^{k+1}(x)^-.$$

Hence the first part of (iii) follows from induction.

By (3.3), (3.4), the first part of (iii), and the fact that $\lim_{x\nearrow u}\lfloor x\rfloor = \lfloor u\rfloor'$ for all $u \in \mathbb{R}$,

$$\lim_{\gamma\nearrow\beta} \varepsilon_n(x,\gamma) = \lim_{\gamma\nearrow\beta} \bigl\lfloor \gamma T_\gamma^{n-1}(x)\bigr\rfloor = \lim_{u\nearrow\beta U_\beta^{n-1}(x)} \lfloor u\rfloor = \bigl\lfloor \beta U_\beta^{n-1}(x)\bigr\rfloor' = \varepsilon_n^*(x,\beta).$$

Therefore, the second part of (iii) follows. □

Definition 3.6 (Beta-shifts). Given a real number $\beta > 1$, define $\pi_\beta\colon I \to \mathcal{B}^{\mathbb{N}}$ by

$$\pi_\beta(x) := \underline{\varepsilon}(x,\beta) = \{\varepsilon_n(x,\beta)\}_{n\in\mathbb{N}},$$

and define $\pi_\beta^*\colon I \to \mathcal{B}^{\mathbb{N}}$ by

$$\pi_\beta^*(x) := \underline{\varepsilon}^*(x,\beta) = \{\varepsilon_n^*(x,\beta)\}_{n\in\mathbb{N}}.$$

Define the *beta-shift* $\mathcal{S}_\beta$ to be the closure in $\mathcal{B}^{\mathbb{N}}$ of the image under π_β of the half-open interval $[0,1)$, in other words,

$$\mathcal{S}_\beta := \overline{\pi_\beta([0,1))}, \tag{3.8}$$

where $\mathcal{B}^{\mathbb{N}}$ is equipped with the product topology. In particular, $\mathcal{S}_\beta$ is closed and $\sigma(\mathcal{S}_\beta) = \mathcal{S}_\beta$, i.e., it is a subshift of the full shift $\mathcal{B}^{\mathbb{N}}$, so we may regard the shift map σ as a self-map $\sigma\colon \mathcal{S}_\beta \to \mathcal{S}_\beta$.

Definition 3.7. Given a real number $\beta > 1$, define X_β to be the closure in $\mathcal{B}^{\mathbb{N}}$ of the image $\pi_\beta(I)$. Define $h_\beta\colon X_\beta \to I$ by

$$h_\beta(\{z_i\}_{i\in\mathbb{N}}) := \sum_{i=1}^{+\infty} z_i \beta^{-i}. \tag{3.9}$$

For each $x \in I$, define $i_x\colon (1,+\infty) \to \mathbb{N}_0^{\mathbb{N}}$ and $i_x^*\colon (1,+\infty) \to \mathbb{N}_0^{\mathbb{N}}$ by

$$i_x(\beta) := \pi_\beta(x) \quad \text{and} \quad i_x^*(\beta) := \pi_\beta^*(x). \tag{3.10}$$

The following lemma shows that our definition of upper β-expansion is equivalent to the definition of incorrect β-expansion in [IT74] and [YT21].

Lemma 3.8. *Fix $\beta > 1$. Then $\pi_\beta(0) = \pi_\beta^*(0) = (0)^\infty$ and $\pi_\beta^*(a) = \lim_{x\nearrow a}\pi_\beta(x)$ for all $a \in (0,1]$.*

Proof. The first part follows from Lemma 3.4 (iii), whereas the second part follows from Lemma 3.4 (iv). □

Proposition 3.9 below collects a number of basic properties of beta-transformations and beta-expansions that will be required later; the majority of the results can be found in the existing literature (specifically, in [Bl89, IT74, Pa60, Re57, YT21]), and for the remainder we provide proofs.

Proposition 3.9. *Given $\beta > 1$, the following statements are true:*

(i) *We have*

$$\pi^*_\beta(1) = \begin{cases} (z_1 z_2 \dots (z_n - 1))^\infty & \text{if } \pi_\beta(1) = z_1 z_2 \dots z_n (0)^\infty, z_n > 0, \\ \pi_\beta(1) & \text{if } \pi_\beta(1) \text{ has infinitely many nonzero terms.} \end{cases}$$

(ii) *For each $x \in (0, 1]$,*

$$\pi^*_\beta(x) = \begin{cases} z_1 z_2 \dots (z_n - 1)\pi^*_\beta(1) & \text{if } \pi_\beta(x) = z_1 z_2 \dots z_n (0)^\infty, z_n > 0, \\ \pi_\beta(x) & \text{if } \pi_\beta(x) \text{ has infinitely many nonzero terms.} \end{cases}$$

(iii) $\sigma \circ \pi_\beta = \pi_\beta \circ T_\beta$ *and* $\sigma \circ \pi^*_\beta = \pi^*_\beta \circ U_\beta$ *on* I.

(iv) $(h_\beta \circ \pi_\beta)(x) = x$ *and* $(h_\beta \circ \pi^*_\beta)(x) = x$ *for each* $x \in I$.

(v) $h_\beta \circ \sigma = T_\beta \circ h_\beta$ *on* $\pi_\beta(I)$ *and* $h_\beta \circ \sigma = U_\beta \circ h_\beta$ *on* $\pi^*_\beta(I)$.

(vi) π_β *and* π^*_β *are strictly increasing, i.e.,* $x < y$ *implies* $\pi_\beta(x) \prec \pi_\beta(y)$ *and* $\pi^*_\beta(x) \prec \pi^*_\beta(y)$.

(vii) $\pi_\beta(x) \prec \pi^*_\beta(y)$ *if* $0 \leqslant x < y \leqslant 1$.

(viii) $\{\omega \in X_\beta : \pi^*_\beta(x) \prec \omega \prec \pi_\beta(x)\} = \emptyset$ *for all* $x \in I$.

(ix) π_β *is right-continuous on* $[0, 1)$ *and* π^*_β *is left-continuous on* $(0, 1]$.

(x) h_β *is a continuous surjection and is non-decreasing, i.e.,* $\omega \prec \omega'$ *implies* $h_\beta(\omega) \leqslant h_\beta(\omega')$.

(xi) *The inverse image* $h^{-1}_\beta(x)$ *of* $x \in (0, 1]$ *consists either of one point* $\pi_\beta(x)$ *or of two points* $\pi_\beta(x)$ *and* $\pi^*_\beta(x)$. *The latter case occurs only when* $T^n_\beta(x) = 0$ *for some* $n \in \mathbb{N}$. *Moreover,* $h^{-1}_\beta(0) = \{(0)^\infty\}$.

(xii) *The function* $h_\beta \colon (X_\beta, d_\beta) \to (I, d)$ *is Lipschitz.*

(xiii) *For each* $x \in (0, 1]$, *the functions* i_x *and* i^*_x *are both strictly increasing functions. Moreover,* $i_0(\beta) = i^*_0(\beta) = (0)^\infty$ *for all* $\beta > 1$.

(xiv) *For each* $x \in I$, *the function* i_x *is right-continuous and the function* i^*_x *is left-continuous.*

Proof. Statements (i), (ii), (ix), and (viii) follow from [YT21, Lemma 1.2].

Statements about π_β in statements (vi) and (iv) follow from [IT74, Proposition 3.2], while statements about π^*_β in (vi) and (iv) follow from [YT21, Lemma 1.2].

Statements (x) and (xi) follow from [IT74, Proposition 3.2].

It remains to prove statements (iii), (v), (vii), and (xii)–(xiv), .

(iii) follows from (3.3), (3.4), and Definition 3.6.

(v) From (iii) and (iv), we have $h_\beta \circ \sigma \circ \pi_\beta = h_\beta \circ \pi_\beta \circ T_\beta = T_\beta = T_\beta \circ h_\beta \circ \pi_\beta$. Thus, $h_\beta \circ \sigma = T_\beta \circ h_\beta$ on $\pi_\beta(I)$. Similarly, we have $h_\beta \circ \sigma = U_\beta \circ h_\beta$ on $\pi^*_\beta(I)$.

(vii) Consider arbitrary $x, y \in I$ with $0 \leqslant x < y \leqslant 1$. By Lemma 3.8, we have $\lim_{z \nearrow y} \pi_\beta(z) = \pi^*_\beta(y)$. Combining this with the fact that π_β is strictly increasing (see (vi)), we get $\pi_\beta(x) \prec \pi_\beta(w) \prec \pi^*_\beta(y)$ for all $w \in (x, y)$.

(xii) Fix $\beta > 1$. Consider a pair of sequences $A = a_1 a_2 \dots$ and $B = b_1 b_2 \dots$ in X_β. Assume that $d_\beta(A, B) = \beta^{-k}$ for some integer $k \in \mathbb{N}$. By (3.9), we have

$$|h_\beta(A) - h_\beta(B)| \leqslant \sum_{n=k}^{+\infty} \frac{|a_n - b_n|}{\beta^n} \leqslant \beta \sum_{n=k}^{+\infty} \frac{1}{\beta^n} = \frac{1}{\beta^{k-2}(\beta - 1)} = \frac{\beta^2}{\beta - 1} d_\beta(A, B).$$

(xiii) By (3.10), Definition 3.6, and Lemma 3.4 (iii), $i_0(\beta) = i_0^*(\beta) = (0)^\infty$ for all $\beta > 1$.

Fix arbitrary $x \in (0, 1]$. It is easy to see that $i_x(\beta_1) \neq i_x(\beta_2)$ when $\beta_1 \neq \beta_2$. So it suffices to prove that i_x is non-decreasing. Assume that there exist $1 < \beta_1 < \beta_2$ such that $i_x(\beta_2) \prec i_x(\beta_1)$. Then there exists $n \in \mathbb{N}$ such that $\varepsilon_k(\beta_1, x) = \varepsilon_k(\beta_2, x)$ for all $k \in \{1, \dots, n-1\}$ and $\varepsilon_n(\beta_1, x) > \varepsilon_n(\beta_2, x)$. Then we have $\varepsilon_n(\beta_1, x) \geqslant \varepsilon_n(\beta_2, x) + 1 \geqslant 1$. By (iii) and (iv), for each $\beta > 1$, we have

$$x = h_\beta(\pi_\beta(x)) = \sum_{k=1}^{n-1} \frac{\varepsilon_k(x, \beta)}{\beta^k} + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \frac{T_\beta^n(x)}{\beta^n}.$$

So we obtain,

$$\begin{aligned} x = \sum_{k=1}^{n-1} \frac{\varepsilon_k(x, \beta_2)}{\beta_2^k} + \frac{\varepsilon_n(x, \beta_2)}{\beta_2^n} + \frac{T_{\beta_2}^n(x)}{\beta_2^n} &\leqslant \sum_{k=1}^{n-1} \frac{\varepsilon_k(x, \beta_2)}{\beta_2^k} + \frac{\varepsilon_n(x, \beta_2) + 1}{\beta_2^n} \\ &\leqslant \sum_{k=1}^{n} \frac{\varepsilon_k(x, \beta_1)}{\beta_2^k} < \sum_{k=1}^{n} \frac{\varepsilon_k(x, \beta_1)}{\beta_1^k} \leqslant x, \end{aligned}$$

which leads to a contradiction. So i_x is strictly increasing. Similarly, we can prove i_x^* is strictly increasing.

(xiv) Fix $x \in I$. By Lemma 3.5 (ii), we have $\varepsilon_n(x, \cdot)$ is right-continuous and $\varepsilon_n^*(x, \cdot)$ is left-continuous for each $n \in \mathbb{N}$. By Definition 3.6 and (3.10), (xiv) follows. ☐

The following classification of values $\beta > 1$, and the interpretation in terms of dynamical behaviour, will be required in our subsequent investigations.

Definition 3.10 (Classification of $\beta > 1$). A real number $\beta > 1$ is said to be

(i) a *simple beta-number* if $\underline{\varepsilon}(1, \beta)$ has only finitely many nonzero terms;

(ii) a *non-simple beta-number* if $\underline{\varepsilon}(1, \beta)$ is preperiodic (i.e., there exists $n \in \mathbb{N}$ such that $\sigma^n(\underline{\varepsilon}(1, \beta))$ is periodic), but β is not a simple beta-number;

(iii) *non-preperiodic* if β is not a *beta-number* (i.e., β satisfies neither (i) nor (ii) above).

Remark 3.11. The terminology *beta-number*, as well as *simple beta-number*, was introduced by Parry [Pa60], who proved (see [Pa60, Theorem 5]) that the set of simple beta-numbers is dense in $(1, +\infty)$. Some authors refer to simple beta-numbers as *Parry numbers* (see e.g. [Ka15]). It is readily seen that β is a simple beta-number if and only if 1 is a periodic point of U_β.

We recall the following notion (see e.g. [PU10, Chapter 4]):

Definition 3.12 (Distance-expanding map). For (X, d) a compact metric space, $T \colon X \to X$ is called a *distance-expanding map* if there exist constants $\lambda > 1$ and $\eta > 0$ such that for all $x, y \in X$ with $d(x, y) < 2\eta$, we have

$$d(T(x), T(y)) \geqslant \lambda d(x, y).$$

The following proposition summarises the relation between periodic points and invariant measures of T_β and U_β.

Proposition 3.13. *If $\beta > 1$, then T_β and U_β satisfy the following properties:*

(i) $T_\beta^{-1}(0) = \{0\} \cup D_\beta$, $T_\beta^{-1}(1) = \emptyset$, $U_\beta^{-1}(0) = \{0\}$, *and* $U_\beta^{-1}(1) = D_\beta$. *Moreover,* T_β *and* U_β *coincide on* $I \smallsetminus D_\beta$.

(ii) $\operatorname{Per}(T_\beta) \subseteq \operatorname{Per}(U_\beta)$. *If* $\mathcal{O}'_\beta$ *is a periodic orbit for* U_β, *then* $\mathcal{O}'_\beta \subseteq \operatorname{Per}(T_\beta)$ *if and only if* $1 \notin \mathcal{O}'_\beta$.

(iii) $\mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$. *If* $\mu \in \mathcal{M}(I, U_\beta)$, *then* $\mu \in \mathcal{M}(I, T_\beta)$ *if and only if* $\mu(\{1\}) = 0$.

(iv) *If* β *is not a simple beta-number, then* $\operatorname{Per}(T_\beta) = \operatorname{Per}(U_\beta)$ *and* $\mathcal{M}(I, T_\beta) = \mathcal{M}(I, U_\beta)$.

(v) *If* β *is a simple beta-number, then* $\operatorname{Per}(U_\beta) = \operatorname{Per}(T_\beta) \cup \mathcal{O}'_\beta(1)$ *and* $\mathcal{M}(I, U_\beta)$ *is the convex hull of* $\bigl\{\mu_{\mathcal{O}'_\beta(1)}\bigr\} \cup \mathcal{M}(I, T_\beta)$.

(vi) T_β *and* U_β *are distance-expanding. Specifically, if* $x, y \in I$ *with* $|x - y| < 1/(2\beta)$, *then* $|T_\beta(x) - T_\beta(y)| \geqslant \beta |x - y|$ *and* $|U_\beta(x) - U_\beta(y)| \geqslant \beta |x - y|$.

Proof. (i) This property follows immediately from the definitions of T_β, U_β, and D_β.

(ii) Fix a periodic orbit $\mathcal{O}_\beta$ of T_β that is not $\{0\}$. By (i) we have $1 \notin \mathcal{O}_\beta$ and $\mathcal{O}_\beta \cap D_\beta = \emptyset$. Then $U_\beta(x) = T_\beta(x)$ for all $x \in \mathcal{O}_\beta$. Hence $\mathcal{O}_\beta$ is also a periodic orbit of U_β.

Fix a periodic orbit $\mathcal{O}'_\beta$ of U_β. If $1 \in \mathcal{O}'_\beta$, by (i) we get that $\mathcal{O}'_\beta$ is not a periodic orbit of T_β. If $1 \notin \mathcal{O}'_\beta$, by (i) we have $\mathcal{O}'_\beta \cap D_\beta = \emptyset$. Then we have $U_\beta(x) = T_\beta(x)$ for all $x \in \mathcal{O}'_\beta$. Hence $\mathcal{O}'_\beta \subseteq \operatorname{Per}(T_\beta)$.

(iii) Fix an arbitrary $\mu \in \mathcal{M}(I, T_\beta)$. By (i) we have $\mu(\{1\}) = \mu\bigl(T_\beta^{-1}(1)\bigr) = \mu(\emptyset) = 0$ and $\mu(D_\beta) = \mu\bigl(T_\beta^{-1}(0)\bigr) - \mu(\{0\}) = 0$. Then we have $\mu\bigl(U_\beta^{-1}(0)\bigr) = \mu(\{0\})$ and $\mu\bigl(U_\beta^{-1}(1)\bigr) = \mu(D_\beta) = 0 = \mu(\{1\})$. By definition we have $T_\beta^{-1}(Y) = U_\beta^{-1}(Y)$ for all Borel measurable subsets $Y \subseteq (0,1)$. Hence, we have $\mu \in \mathcal{M}\bigl(I, U_\beta\bigr)$.

Now fix an arbitrary $\nu \in \mathcal{M}\bigl(I, U_\beta\bigr)$. If $\nu(\{1\}) = 0$, by (i) we have $\nu(D_\beta) = \nu\bigl(U_\beta^{-1}(1)\bigr) = \nu(\{1\}) = 0$ and $\nu\bigl(T_\beta^{-1}(0)\bigr) = \nu(\{0\}) + \nu(D_\beta) = \nu(\{0\})$. By definition we have $T_\beta^{-1}(Y) = U_\beta^{-1}(Y)$ for all Borel measurable subsets $Y \subseteq (0,1)$. Hence, we have $\nu \in \mathcal{M}(I, T_\beta)$. If on the other hand $\nu(\{1\}) > 0$, since $T_\beta^{-1}(1) = \emptyset$ by (i), we have $\nu \notin \mathcal{M}(I, T_\beta)$.

(iv) and (v) Recall that 1 is a periodic point of U_β if and only if β is a simple beta-number (see Remark 3.11). Then the first part of (iv) and the first part of (v) follow immediately from (ii).

Fix an arbitrary $\mu \in \mathcal{M}(I, U_\beta)$. If β is not a simple beta-number, then 1 is not a periodic point of U_β (see Remark 3.11) and it is straightforward to check that $\mu(\{1\}) = 0$. By (iii), $\mu \in \mathcal{M}(I, T_\beta)$. The second part of (iv) follows.

Assume that β is a simple beta-number. Since $\mathcal{O}'_\beta(1)$ is a periodic orbit of U_β, then it is easy to see that $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in \mathcal{O}'_\beta(1)$. Write $t \coloneqq \mu(\mathcal{O}'_\beta(1))$. When $t = 1$, we have $\mu(\{x\}) = 1/\operatorname{card} \mathcal{O}'_\beta(1)$ for all $x \in \mathcal{O}'_\beta(1)$. In this case, $\mu = \mu_{\mathcal{O}'_\beta(1)}$. When $t \in [0,1)$, let us write $\nu \coloneqq \frac{1}{1-t}\bigl(\mu - t\mu_{\mathcal{O}'_\beta(1)}\bigr)$. Then $\nu \in \mathcal{M}(I, U_\beta)$ and $\nu(\{1\}) = 0$. By (iii), $\nu \in \mathcal{M}(I, T_\beta)$. In this case, $\mu = t\mu_{\mathcal{O}'_\beta(1)} + (1-t)\nu$. The second part of (v) follows.

(vi) Fix arbitrary $x, y \in I$ with $0 < y - x < 1/(2\beta)$. If there exists an integer i such that $i/\beta \leqslant x < y < (i+1)/\beta$, then $T_\beta(y) - T_\beta(x) = \beta(y - x)$. Otherwise, there exists an integer i such that $(i-1)/\beta < x < i/\beta \leqslant y < (i+1)/\beta$. Then we have $T_\beta(y) - T_\beta(x) = \beta(y - x) - 1 < -1/2 < -\beta|y - x|$. Similarly, we can prove $U_\beta(y) - U_\beta(x) \geqslant \beta(y - x)$. □

While the support of any T-invariant probability measure μ satisfies $T(\operatorname{supp}\mu) = \operatorname{supp}\mu$ in the case that T is continuous (see e.g. [Ak93, p. 156]), the same is not true for the discontinuous maps T_β and U_β, nevertheless we do have the following result.

Lemma 3.14. *Suppose $\beta > 1$ and $\mu \in \mathcal{M}(I, U_\beta)$. Then $U_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$ if $0 \notin \operatorname{supp}\mu$, and $T_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$ if $1 \notin \operatorname{supp}\mu$.*

Proof. Let us write $\mathcal{K} \coloneqq \operatorname{supp}\mu$.

Assume that $0 \notin \mathcal{K}$ and denote $\delta_1 \coloneqq d(\mathcal{K}, 0) > 0$. By (3.2), for each $y \in D_\beta$, we obtain

$$\mu((y, y + \delta_1/\beta) \cap I) \leqslant \mu\bigl(U_\beta^{-1}(0, \delta_1)\bigr) = \mu((0, \delta_1)) = 0.$$

So $\mathcal{K} \cap (y, y + \delta_1/\beta) = \emptyset$ for each $y \in D_\beta$. Hence for each pair of x, $y \in \mathcal{K}$ with $|x - y| < \delta_1/\beta$, we have $(x, y) \cap D_\beta = \emptyset$ and $U_\beta(x) - U_\beta(y) = \beta(x - y)$. So $U_\beta|_{\mathcal{K}}$ is continuous and μ can be seen as an invariant measure for $(\mathcal{K}, U_\beta|_{\mathcal{K}})$. Therefore, $U_\beta(\mathcal{K}) = \mathcal{K}$ ([Ak93, p. 156]).

Assume that $1 \notin \mathcal{K}$ and denote $\delta_2 \coloneqq d(\mathcal{K}, 1) > 0$. By Proposition 3.13 (iii), $\mu \in \mathcal{M}(I, T_\beta)$. By (3.1), for each $y \in D_\beta$, we obtain

$$\mu((y - \delta_2/\beta, y) \cap I) \leqslant \mu\bigl(T_\beta^{-1}(0, \delta_2)\bigr) = \mu((0, \delta_2)) = 0.$$

So $\mathcal{K} \cap (y - \delta_2/\beta, y) = \emptyset$ for each $y \in D_\beta$. Hence for each pair of x, $y \in \mathcal{K}$ with $|x - y| < \delta_2/\beta$, we have $(x, y) \cap D_\beta = \emptyset$ and $T_\beta(x) - T_\beta(y) = \beta(x - y)$. So $T_\beta|_{\mathcal{K}}$ is continuous and μ can be seen as an invariant measure for $(\mathcal{K}, T_\beta|_{\mathcal{K}})$. Therefore, $T_\beta(\mathcal{K}) = \mathcal{K}$ ([Ak93, p. 156]). □

3.2. **Monotonicity and approximation properties in parameter space.** Here we recall some monotonicity and approximation properties for the one-parameter family of beta-shifts.

The following proposition characterises those sequences on the alphabet $\mathcal{B} = \{0, 1, \dots, \lfloor\beta\rfloor\}$ that arise as the β-expansion of a real number $x \in [0, 1]$.

Proposition 3.15. *Given $\beta > 1$, the following statements are true:*

(i) $\pi_\beta([0, 1)) = \bigl\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \prec \pi^*_\beta(1) \text{ for all } n \in \mathbb{N}_0\bigr\}$.

(ii) *$\mathcal{S}_\beta$ can also be expressed as*

$$\mathcal{S}_\beta = \bigl\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \preceq \pi^*_\beta(1) \text{ for all } n \in \mathbb{N}_0\bigr\}. \tag{3.11}$$

Proof. (i) follows from [Pa60, Theorem 3] and Proposition 3.9 (viii), while (ii) is exactly [IT74, Lemma 4.4]. □

Remark 3.16. (i) Define $\widetilde{X}_\beta \coloneqq \bigl\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \preceq \pi_\beta(1) \text{ for all } n \in \mathbb{N}_0\bigr\}$.

(ii) If β is not a simple beta-number, then $\pi_\beta(1) = \pi^*_\beta(1)$ (see Proposition 3.9 (i)), and hence $\mathcal{S}_\beta = X_\beta = \widetilde{X}_\beta$ by (3.11), together with Definitions 3.6 and 3.7.

(iii) If β is a simple beta-number, then $\pi^*_\beta(1) \prec \pi_\beta(1)$ (see Proposition 3.9 (i)), hence X_β is the union of the beta-shift $\mathcal{S}_\beta$ and the singleton set $\{\pi_\beta(1)\}$ (which is disjoint from $\mathcal{S}_\beta$), and

$$\mathcal{S}_\beta \subseteq X_\beta \subseteq \widetilde{X}_\beta \tag{3.12}$$

by (3.11), Definitions 3.6 and 3.7. The inclusions in (3.12) are proper: for example, when $\beta = 2$, we have

$$2(0)^\infty \in X_2 \smallsetminus \mathcal{S}_2, \quad 12(0)^\infty \in \widetilde{X}_2 \smallsetminus X_2.$$

(iv) If β is a simple beta-number then σ maps $\mathcal{S}_\beta$ surjectively onto itself, and maps $\widetilde{X}_\beta$ surjectively onto itself, but $\sigma \colon X_\beta \to X_\beta$ is not surjective.

(v) A complement to Proposition 3.15 (i) is that, for each $A \in \mathbb{N}_0^{\mathbb{N}}$, there exists $\beta > 1$ with $A = \underline{\varepsilon}(1,\beta)$ if and only if $\sigma^n(A) \prec A$ for all $n \in \mathbb{N}$; and if such a number $\beta > 1$ exists then it is unique (see [Pa60, Corollary 1]). Consequently, each of the three classes in Definition 3.10 is readily seen to be non-empty.

(vi) Some authors define the beta-shift to be either X_β or $\widetilde{X}_\beta$, instead of $\mathcal{S}_\beta$. For example it is defined to be X_β in [AB07, p. 1696], [Sc97, Definition 2.2], and [KQ22, p. 1438]), and defined to be $\widetilde{X}_\beta$ in [Si76, p. 248] and [Wa82, p. 179].

Lemma 3.17. *Given $\beta > 1$, the following statements are true:*

(i) *If $1 < \beta' < \beta$, then $\mathcal{S}_{\beta'} \subseteq \mathcal{S}_\beta$.*

(ii) $\mathcal{S}_\beta = \overline{\bigcup_{\gamma \in (1,\beta)} \mathcal{S}_\gamma}$.

Proof. (i) Assume that $1 < \beta' < \beta$. By Proposition 3.9 (xiii) and (3.10), we have $\pi^*_{\beta'}(1) \prec \pi^*_\beta(1)$. By (3.11), $\mathcal{S}_{\beta'} \subseteq \mathcal{S}_\beta$.

(ii) Assume that $\pi^*_\beta(1) = a_1a_2\dots$. For each $n \in \mathbb{N}$, put $A_n \coloneqq a_1 \dots a_n 00\dots$. By Proposition 3.9 (xiv), we get that $\pi^*_\beta(1) = \lim_{\gamma \nearrow \beta} \pi^*_\gamma(1)$. Thus, for each $n \in \mathbb{N}$, there exists $\gamma_n \in (1,\beta)$ such that $A_n \preceq \pi^*_{\gamma_n}(1)$. Fix arbitrary $B = b_1b_2\dots \in \mathcal{S}_\beta$. Put $B_n \coloneqq b_1 \dots b_n 00\dots$ for each $n \in \mathbb{N}$. By (3.11), for each $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we have $\sigma^k(B_n) \preceq A_n \preceq \pi^*_{\gamma_n}(1)$. Thus, $B_n \in \mathcal{S}_{\gamma_n}$. Note that $\lim_{n \to +\infty} B_n = B$, so $B \in \overline{\bigcup_{\gamma \in (1,\beta)} \mathcal{S}_\gamma}$. Since B is chosen arbitrarily, we obtain from (i) that $\mathcal{S}_\beta = \overline{\bigcup_{\gamma \in (1,\beta)} \mathcal{S}_\gamma}$. $\square$

Remark 3.18. Lemma 3.17 is hinted at as part of [IT74, Proposition 4.1] (though in [IT74] it is slightly mis-stated, and not proved, so for the convenience of the reader we include a proof here). We note that another part of [IT74, Proposition 4.1] is false: in general it is not the case that $\mathcal{S}_\beta = \bigcap_{\gamma > \beta} \mathcal{S}_\gamma$ (for example if $\beta = 2$ then $2(0)^\infty \in \mathcal{S}_\gamma$ for all $\gamma > 2$, but $2(0)^\infty \notin \mathcal{S}_2$), however the intersection can be expressed as

$$\widetilde{X}_\beta = \bigcap_{\gamma > \beta} \mathcal{S}_\gamma.$$

Definition 3.19. For $1 < \gamma < \beta$, define

$$H^\gamma_\beta \coloneqq h_\beta(\mathcal{S}_\gamma) = \Biggl\{ \sum_{i=1}^{+\infty} z_i \beta^{-i} : \{z_i\}_{i \in \mathbb{N}} \in \mathcal{S}_\gamma \Biggr\}, \tag{3.13}$$

and if $\psi \in C^{0,\alpha}(I)$ then define the corresponding *restricted maximum ergodic average*

$$Q_{\beta,\gamma}(\psi) \coloneqq Q\bigl(T_\beta|_{H^\gamma_\beta}, \psi|_{H^\gamma_\beta}\bigr) = \sup\biggl\{ \int \psi \, \mathrm{d}\mu : \mu \in \mathcal{M}(I, T_\beta), \operatorname{supp} \mu \subseteq H^\gamma_\beta \biggr\}. \tag{3.14}$$

Lemma 3.20. *Suppose $\beta > 1$. If $\mathcal{K} \subseteq I$ is a non-empty compact set with $1 \notin \mathcal{K} = T_\beta(\mathcal{K})$, then there exists $\beta' \in (1,\beta)$ such that $\mathcal{K} \subseteq H^\gamma_\beta$ for each $\gamma \in (\beta',\beta)$.*

Similarly, if $\mathcal{K}'$ is a non-empty compact set with $1 \notin \mathcal{K}' = U_\beta(\mathcal{K}')$, then there exists $\beta' \in (1,\beta)$ such that $\mathcal{K}' \subseteq H^\gamma_\beta$ for each $\gamma \in (\beta',\beta)$.

Proof. If $\mathcal{K}$ is a non-empty compact set, with $1 \notin \mathcal{K} = T_\beta(\mathcal{K})$, then the largest point in $\mathcal{K}$ is strictly smaller than 1. By Proposition 3.9 (vi), (xiv), and (3.10), there exists $\beta' \in (1,\beta)$ such that $\max\{\pi^*_\beta(x) : x \in \mathcal{K}\} \prec \pi^*_{\beta'}(1)$. Furthermore, by Proposition 3.9 (vi), (viii), and the fact that $\pi^*_{\beta'}(1) \in X_\beta$ (see (3.11) and Lemma 3.17 (i)), we have $\max\{\pi_\beta(x) : x \in \mathcal{K}\} \preceq \pi^*_{\beta'}(1)$. By Proposition 3.9 (iii), we get $\sigma(\pi_\beta(\mathcal{K})) = \pi_\beta(\mathcal{K})$. So if $z \in \mathcal{K}$ then $\sigma^n(\pi_\beta(z)) \preceq \pi^*_{\beta'}(1)$

for all $n \in \mathbb{N}_0$, and therefore by (3.11), $\pi_\beta(z) \in \mathcal{S}_{\beta'}$. Hence, by the definition of H_β^γ and Proposition 3.9 (iv), we have $\mathcal{K} = h_\beta(\pi_\beta(\mathcal{K})) \subseteq h_\beta(\mathcal{S}_{\beta'}) = H_\beta^{\beta'}$. By (3.13) and Lemma 3.17 (i), we have $H_\beta^{\beta'} \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$. Therefore, $\mathcal{K} \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$.

Now let $\mathcal{K}'$ be a non-empty compact set with $U_\beta(\mathcal{K}') = \mathcal{K}'$. Applying Proposition 3.13 (i), we have $\mathcal{K}' \cap D_\beta = \emptyset$, so $T_\beta(x) = U_\beta(x)$ for each $x \in \mathcal{K}'$. Thus, $T_\beta(\mathcal{K}') = \mathcal{K}'$. Therefore, there exists $\beta' \in (1, \beta)$ such that $\mathcal{K}' \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$. □

Recall that a homeomorphism $g\colon X_1 \to X_2$ between metric spaces (X_1, d_1) and (X_2, d_2) is *bi-Lipschitz* if there exists a constant $C \geqslant 1$ such that for all u, $v \in X_1$,

$$C^{-1} d_1(u, v) \leqslant d_2(g(u), g(v)) \leqslant C d_1(u, v).$$

Lemma 3.21. *For $1 < \gamma < \beta$, the map $\pi_\beta|_{H_\beta^\gamma}\colon \left(H_\beta^\gamma, d\right) \to (\mathcal{S}_\gamma, d_\beta)$ is bi-Lipschitz.*

Proof. Define $\delta \coloneqq d\left(H_\beta^\gamma, 1\right)$. By (3.10) and Proposition 3.9 (i) and (xiii), we have $\pi_\gamma^*(1) \prec \pi_\beta^*(1) \preceq \pi_\beta(1)$. Hence $\pi_\beta(1), \pi_\beta^*(1) \notin \mathcal{S}_\gamma$ (see (3.11)). So by Proposition 3.9 (x), (xi), and (3.13), we have $1 \notin H_\beta^\gamma$ and $0 < \delta \leqslant 1$.

Assume that $x, y \in H_\beta^\gamma$ satisfy $d_\beta(\pi_\beta(x), \pi_\beta(y)) = \beta^{-n}$ and $x < y$, then

$$\pi_\beta(x) = a_1 \dots a_{n-1} b_n b_{n+1} \dots, \quad \pi_\beta(y) = a_1 \dots a_{n-1} c_n c_{n+1} \dots,$$

where $b_n < c_n$. Then by Proposition 3.9 (iv) and the definition of h_β,

$$d(x, y) = d(h_\beta(\pi_\beta(x)), h_\beta(\pi_\beta(y))) = \beta^{-n+1} d(h_\beta(b_n b_{n+1} \dots), h_\beta(c_n c_{n+1} \dots)) \leqslant \beta^{-n+1}. \tag{3.15}$$

Moreover, by the definition of h_β and H_β^γ, we have

$$\begin{aligned} h_\beta(b_n b_{n+1} \dots) &= (b_n + h_\beta(b_{n+1} \dots))/\beta \leqslant (b_n + 1 - \delta)/\beta \quad \text{and} \\ h_\beta(c_n c_{n+1} \dots) &= (c_n + h_\beta(c_{n+1} \dots))/\beta \geqslant c_n/\beta \geqslant (b_n + 1)/\beta. \end{aligned}$$

So we have $d(h_\beta(b_n b_{n+1} \dots), h_\beta(c_n c_{n+1} \dots)) \geqslant \delta/\beta$. Thus, by (3.15), we have

$$d(x, y) \geqslant \delta \beta^{-n} = \delta d_\beta(\pi_\beta(x), \pi_\beta(y)). \tag{3.16}$$

Let us write $C \coloneqq \max\{\beta, 1/\delta\} \geqslant 1$. Combining (3.15) and (3.16), we have

$$C^{-1} d(x, y) \leqslant d_\beta(\pi_\beta(x), \pi_\beta(y)) \leqslant C d(x, y).$$ □

Lemma 3.22. *For each $\beta > 1$ and each $\gamma \in (1, \beta)$, the set H_β^γ is a closed subset of I satisfying $T_\beta(H_\beta^\gamma) \subseteq H_\beta^\gamma$ and the restricted beta-transformation $T_\beta|_{H_\beta^\gamma}\colon H_\beta^\gamma \to H_\beta^\gamma$ has the following properties:*

(i) *$T_\beta|_{H_\beta^\gamma}$ is Lipschitz.*

(ii) *$T_\beta|_{H_\beta^\gamma}$ is distance-expanding.*

(iii) *If γ is a simple beta-number, then $T_\beta|_{H_\beta^\gamma}$ is open.*

Proof. By (3.11), $\mathcal{S}_\gamma$ is closed and $\sigma(\mathcal{S}_\gamma) \subseteq \mathcal{S}_\gamma$. By Proposition 3.9 (iv), (3.13), and Lemma 3.21, $\pi_\beta|_{H_\beta^\gamma}$ is bi-Lipschitz with inverse $h_\beta|_{\mathcal{S}_\gamma}$ and H_β^γ is closed. By Proposition 3.9 (iii), $T_\beta(H_\beta^\gamma) \subseteq H_\beta^\gamma$. Since $T_\beta|_{H_\beta^\gamma} = h_\beta|_{\mathcal{S}_\gamma} \circ \sigma|_{\mathcal{S}_\gamma} \circ \pi_\beta|_{H_\beta^\gamma}$ (see Proposition 3.9 (iii) and (v)), we obtain (i).

(ii) follows from Proposition 3.13 (vi).

To verify (iii), assume that γ is a simple beta-number. Then $(\mathcal{S}_\gamma, \sigma)$ is a subshift of finite type (see e.g. [Bl89, Proposition 4.1]), and therefore $\sigma|_{\mathcal{S}_\gamma}$ is open (see e.g. [URM22, Theorem 3.2.12]). By Lemma 3.21 and Proposition 3.9 (iv), we know that $\pi_\beta|_{H_\beta^\gamma}$ and $h_\beta|_{\mathcal{S}_\gamma}$ are homeomorphisms. It follows that $T_\beta|_{H_\beta^\gamma} = h_\beta|_{\mathcal{S}_\gamma} \circ \sigma|_{\mathcal{S}_\gamma} \circ \pi_\beta|_{H_\beta^\gamma}$ is open. $\square$

3.3. **Cylinders.**

Definition 3.23. Fix $\beta > 1$ and $n \in \mathbb{N}$. A length-n prefix $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ is said to be *β-admissible* if $\epsilon_1 \dots \epsilon_n (0)^\infty \in \pi_\beta([0,1))$.

For each β-admissible length-n prefix, we define the corresponding *n-cylinder* to be

$$I(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = \{x \in [0,1) : \varepsilon_i(x, \beta) = \epsilon_i \text{ for all } 1 \leqslant i \leqslant n\}, \tag{3.17}$$

and if $T_\beta^n(I(\epsilon_1, \epsilon_2, \dots, \epsilon_n)) = [0,1)$ we say that the cylinder $I(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ is *full.* Let W^n denote the set of all n-cylinders, and let W_0^n denote the set of all full n-cylinders.

Note that the n-cylinder $I(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ is a left-closed and right-open interval, with the left endpoint

$$\frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \dots + \frac{\epsilon_n}{\beta^n}. \tag{3.18}$$

For each $n \in \mathbb{N}$ and each $I^n \in W^n$, let $\overline{I}^n$ denote the closure of I^n. Denote $\widetilde{I}^n$ to be $\overline{I}^n$ excluding its left endpoint. If $m \in \{1, \dots, n\}$ and $I^n \in W^n$, we write $T_{\beta,I^n}^m \colon \overline{I}^n \to I$ for the continuous extension of $T_\beta^m|_{I^n}$ to $\overline{I}^n$. Moreover, for each function $\phi \colon I \to \mathbb{R}$, we define

$$S_{m,I^n}\phi \coloneqq \sum_{i=0}^{m-1} \phi \circ T_{\beta,I^n}^i. \tag{3.19}$$

Proposition 3.24. *Fix $\beta > 1$, $n \in \mathbb{N}$, and $I^n \coloneqq I(\epsilon_1, \dots, \epsilon_n) \in W^n$. For each $i \in \{1, \dots, n-1\}$, denote $I^{n-i} \coloneqq I(\epsilon_{i+1}, \dots, \epsilon_n)$.*

(i) *For each $m \in \{1, \dots, n-1\}$, $(\epsilon_{m+1}, \dots, \epsilon_n)$ is a β-admissible prefix and $T_\beta^m(I^n) \subseteq I^{n-m}$.*

(ii) *If $m \in \{1, \dots, n\}$ and $x, y \in \overline{I}^n$, then $T_{\beta,I^n}^m(y) - T_{\beta,I^n}^m(x) = \beta^m(y-x)$. Consequently, T_{β,I^n}^m is continuous and strictly increasing.*

(iii) *If $\epsilon_n > 0$, then $I(\epsilon_1, \dots, \epsilon_{n-1}, b) \in W_0^n$ for $b \in \{0, \dots, \epsilon_n - 1\}$ with the right endpoint $\sum_{i=1}^{n-1} \epsilon_i/\beta^i + (b+1)/\beta^n$.*

(iv) *If $I^n \in W_0^n$, then there is a T_{β,I^n}^n-fixed point in $\overline{I}^n$.*

(v) *There exists $m \in \{0, 1, \dots, n\}$ such that $T_{\beta,I^n}^n(\overline{I}^n) = [0, U_\beta^m(1)]$.*

(vi) *$T_{\beta,I^n}^{k+m} = T_{\beta,I^{n-m}}^k \circ T_{\beta,I^n}^m$ on $\overline{I}^n$ for each $m \in \{1, \dots, n-1\}$ and each $k \in \{1, \dots, n-m\}$.*

(vii) *For each $m \in \{1, \dots, n\}$, $T_{\beta,I^n}^m = U_\beta^m$ on $\widetilde{I}^n$.*

(viii) *$[0,1) = \bigsqcup_{I^n \in W^n} I^n$ and $(0,1] = \bigsqcup_{I^n \in W^n} \widetilde{I}^n$, where $\sqcup$ means the union of disjoint sets.*

(ix) *$T_\beta^{-n}(0) \smallsetminus \{1\} = \{T_{\beta,I^n}^{-n}(0) : I^n \in W^n\}$ and*

$$U_\beta^{-n}(x) = \{T_{\beta,I^n}^{-n}(x) : x \in T_{\beta,I^n}(\overline{I}^n),\ I^n \in W^n\}$$

for each $x \in (0,1]$.

Proof. Denote $\pi_\beta^*(1) = a_1 a_2 \dots$ in this proof. Recall that $(\pi_\beta \circ T_\beta)(x) = (\sigma \circ \pi_\beta)(x)$ and $(h_\beta \circ \pi_\beta)(x) = x$ for each $x \in I$ (see Proposition 3.9 (iii) and (iv)). By (3.17) and Proposition 3.9 (vi), $x \in I^n$ if and only if

$$\epsilon_1\epsilon_2 \dots \epsilon_n (0)^\infty \preceq \pi_\beta(x) \prec \epsilon_1\epsilon_2 \dots (\epsilon_n + 1)(0)^\infty. \tag{3.20}$$

(i) Fix arbitrary $x \in I^n$. Then (3.20) holds. For each $m \in \{1, \ldots, n-1\}$, we have $\pi_\beta(T_\beta^m(x)) = \sigma^m(\pi_\beta(x))$. Hence $\epsilon_{m+1}\ldots\epsilon_n(0)^\infty \preceq \pi_\beta(T_\beta^m(x)) \prec \epsilon_{m+1}\ldots(\epsilon_n+1)(0)^\infty$, and thus $T_\beta^m(x) \in I^{n-m}$. By the arbitrariness of $x \in I^n$, $T_\beta^m(I^n) \subseteq I^{n-m}$.

(ii) For arbitrary $x, y \in I^n$, assume that $\pi_\beta(x) = \epsilon_1 \ldots \epsilon_n x_1 x_2 \ldots$ and $\pi_\beta(y) = \epsilon_1 \ldots \epsilon_n y_1 y_2 \ldots$. So $y - x = h_\beta(\pi_\beta(y)) - h_\beta(\pi_\beta(x)) = \beta^{-n}\sum_{i=1}^{+\infty}\beta^{-i}(y_i - x_i)$ and by Proposition 3.9 (iii) and (v),

$$\begin{aligned} T_\beta^m(y) - T_\beta^m(x) &= h_\beta(\sigma^m(\pi_\beta(y))) - h_\beta(\sigma^m(\pi_\beta(x))) \\ &= \beta^{m-n}\sum_{i=1}^{+\infty}\beta^{-i}(y_i - x_i) = \beta^m(y - x). \end{aligned}$$

Since T^m_{β,I^n} is the continuous extension of $T^m_\beta|_{I^n}$ to $\overline{I}^n$, then (ii) follows.

(iii) Fix arbitrary $b \in \{0, \ldots, \epsilon_n - 1\}$. Write $y_b \coloneqq \sum_{i=1}^{n-1}\epsilon_i/\beta^i + b/\beta^n$ and $z_b \coloneqq \sum_{i=1}^{n-1}\epsilon_i/\beta^i + (b+1)/\beta^n$. Since $(\epsilon_1, \ldots, \epsilon_n)$ is β-admissible and $b + 1 \leqslant \epsilon_n$, we have $y_b < z_b \leqslant 1$. Then $\pi_\beta(y_b) = \epsilon_1 \ldots \epsilon_{n-1} b(0)^\infty$ and $\pi_\beta(z_b) = \epsilon_1 \ldots \epsilon_{n-1}(b+1)(0)^\infty$. Since π_β is strictly increasing (see Proposition 3.9 (vi)), $I(\epsilon_1, \ldots, \epsilon_{n-1}, b) = [y_b, z_b)$ and $\operatorname{diam} I(\epsilon_1, \ldots, \epsilon_{n-1}, b) = \beta^{-n}$. Moreover, $T^n_\beta(I(\epsilon_1, \ldots, \epsilon_{n-1}, b)) = [0, 1)$ by (ii). Therefore $I(\epsilon_1, \ldots, \epsilon_{n-1}, b) \in W_0^n$ with the right endpoint z_b.

(iv) By (ii), T^n_{β,I^n} is continuous. Since I^n is a full cylinder, then $T^n_{\beta,I^n}\bigl(\overline{I}^n\bigr) = \overline{T^n_{\beta,I^n}(I^n)} = \overline{T^n_\beta(I^n)} = I \supseteq \overline{I}^n$, so by the intermediate value theorem, T^n_{β,I^n} has a fixed point.

(v) Denote

$$m \coloneqq \max(\{j \in \mathbb{N} : \epsilon_{n-j+i} = a_i \text{ for all } 1 \leqslant i \leqslant j\} \cup \{0\}).$$

Let y be the left endpoint of I^n and $z \coloneqq h_\beta(A)$ with

$$A \coloneqq \epsilon_1 \ldots \epsilon_{n-m}\pi_\beta^*(1) = \epsilon_1 \ldots \epsilon_n a_{m+1} a_{m+2} \ldots. \tag{3.21}$$

We first check that $A \in \mathcal{S}_\beta$. Fix $k \in \mathbb{N}_0$ arbitrarily. If $k < n - m$, then $\epsilon_{k+1} \ldots, \epsilon_n(0)^\infty \prec a_1 \ldots a_{n-k}(0)^\infty$ by the maximality of m and Proposition 3.9 (vi). So $\sigma^k(A) \prec \pi^*_\beta(1)$. If $k \geqslant n - m$, then $\sigma^k(A) = \sigma^{k-(n-m)}\bigl(\pi^*_\beta(1)\bigr) \preceq \pi^*_\beta(1)$ by (3.21) and Proposition 3.9 (vi). We obtain $A \in \mathcal{S}_\beta$ by (3.11). Since $h_\beta(A) = z$ and $A = \epsilon_1 \ldots \epsilon_{n-m}\pi^*_\beta(1)$, by Proposition 3.9 (ii), (iv), and (xi), $A = \pi^*_\beta(z)$.

Consider arbitrary $x \in I^n$, then $\pi_\beta(x) \in \mathcal{S}_\beta$ by Definition 3.6. So $\sigma^{n-m}(\pi_\beta(x)) \preceq \pi^*_\beta(1) = \sigma^{n-m}(A)$ by Proposition 3.15 (ii) and (3.21). Combining this with the fact that the first $n-m$ terms of $\pi_\beta(x)$ and A coincide (see (3.20)), we get $\pi_\beta(x) \preceq A$. Since h_β is non-decreasing (see Proposition 3.9 (x)), $x \leqslant z = h_\beta(A)$ for all $x \in I^n$ by Proposition 3.9 (vi). Consider arbitrary $w \in [y, z)$. By Proposition 3.9 (vii), $\pi_\beta(w) \preceq \pi^*_\beta(z) = A \preceq \pi_\beta(z)$. Hence $w \in I^n$ for all $w \in [y, z)$ by (3.20).

By our discussion above, z is the right endpoint of I^n. Moreover, by (ii) and Proposition 3.9 (iv), (v), $T^n_{\beta,I^n}\bigl(\overline{I}^n\bigr)$ is a closed interval with the left endpoint 0 and the right endpoint $\beta^n(z - y) = T^m_\beta\bigl(h_\beta\bigl(\pi^*_\beta(1)\bigr)\bigr) = h_\beta\bigl(\sigma^m\bigl(\pi^*_\beta(1)\bigr)\bigr) = U^m_\beta(1)$.

(vi) Fix $m \in \{1, \ldots, n-1\}$ and $k \in \{1, \ldots, n-m\}$ arbitrarily. By (i), we have $T^m_\beta(I^n) \subseteq I^{m-n}$. Thus, we have $T^{k+m}_{\beta,I^n} = T^k_{\beta,I^{m-n}} \circ T^m_{\beta,I^n}$ on I^n since $T^{k+m}_{\beta,I^n} = T^{k+m}_\beta$, $T^m_{\beta,I^n} = T^m_\beta$ on I^n and $T^k_{\beta,I^{m-n}} = T^k_\beta$ on $I^{m-n}(\supseteq T^m_\beta(I^n))$. For y the right endpoint of I^n, and arbitrary $x \in I^n$, by

(ii) we have

$$\begin{aligned} T^{k+m}_{\beta,I^n}(y) - T^{k+m}_{\beta,I^n}(x) &= \beta^{k+m}(y-x) \quad \text{and} \\ \bigl(T^k_{\beta,I^{n-m}} \circ T^m_{\beta,I^n}\bigr)(y) - \bigl(T^k_{\beta,I^{n-m}} \circ T^m_{\beta,I^n}\bigr)(x) &= T^k_{\beta,I^{n-m}}(x + \beta^m(y-x)) - T^k_{\beta,I^{n-m}}(x) \\ &= \beta^{k+m}(y-x). \end{aligned}$$

Therefore, (vi) holds.

(vii) Fix $m \in \{1, \dots, n\}$ and let $y \in \widetilde{I}^n$. By (ii) and Lemma 3.4 (iv), we obtain $T^m_{\beta,I^n}(y) = \lim_{x \nearrow y} T^m_{\beta,I^n}(x) = \lim_{x \nearrow y} T^m_\beta(x) = U^m_\beta(y)$.

(viii) By (3.17) and Proposition 3.9 (vi), $I^n_1 \cap I^n_2 = \emptyset$ for each I^n_1, $I^n_2 \in W^n$ with $I^n_1 \neq I^n_2$. For each $x \in [0,1)$, assume $\pi_\beta(x) = a_1a_2 \dots a_n \dots$. Then $x \in I(a_1, \dots, a_n)$. So $[0,1) = \bigsqcup_{I^n \in W^n} I^n$. Since I^n is a left-closed and right-open interval and $\widetilde{I}^n$ is defined to be $\overline{I}^n$ excluding its left endpoint, it follows immediately that the second identity in (viii) holds.

(ix) Consider arbitrary $y_1 \in T^{-n}_\beta(0) \smallsetminus \{1\}$. There exists a unique $I^n \in W^n$ satisfying $y_1 \in I^n$ by (viii). So $T^n_{\beta,I^n}(y_1) = T^n_\beta(y_1) = 0$. Hence by the arbitrariness of $y_1 \in T^{-n}_\beta(0) \smallsetminus \{1\}$,

$$T^{-n}_\beta(0) \smallsetminus \{1\} \subseteq \bigl\{T^{-n}_{\beta,I^n}(0) : I^n \in W^n\bigr\}.$$

Fix arbitrary $I^n \in W^n$. By (v), $0 \in T^n_{\beta,I^n}\bigl(\overline{I}^n\bigr)$. Write $y_2 \coloneqq T^{-n}_{\beta,I^n}(0)$ by (ii). Then y_2 is the left endpoint of I^n by (ii) and (v). Moreover, $y_2 \neq 1$ and $T^n_\beta(y_2) = T^n_{\beta,I^n}(y_2) = 0$. Hence by the arbitrariness of $I^n \in W^n$,

$$T^{-n}_\beta(0) \smallsetminus \{1\} \supseteq \bigl\{T^{-n}_{\beta,I^n}(0) : I^n \in W^n\bigr\}.$$

The first part of (ix) follows.

Fix $x \in (0,1]$. Consider arbitrary $z_1 \in U^{-n}_\beta(x)$. Since 0 is a fixed point of U_β (see (3.2)), we get $z_1 \in (0,1]$. There exists a unique $I^n \in W^n$ satisfying $z_1 \in \widetilde{I}^n$ by (viii). By (vii), $T^n_{\beta,I^n}(z_1) = U^n_\beta(z_1) = x$. Hence by the arbitrariness of $z_1 \in U^{-n}_\beta(x)$,

$$U^{-n}_\beta(x) \subseteq \bigl\{T^{-n}_{\beta,I^n}(x) : x \in T_{\beta,I^n}\bigl(\overline{I}^n\bigr),\, I^n \in W^n\bigr\}.$$

Fix arbitrary $I^n \in W^n$ satisfying $x \in T^n_{\beta,I^n}\bigl(\overline{I}^n\bigr)$. Write $z_2 \coloneqq T^{-n}_{\beta,I^n}(x)$ by (ii). Then since $x \in (0,1]$, z_2 is not the left endpoint of $\overline{I}^n$ by (ii) and (v). Moreover, $U^n_\beta(z_2) = T^n_{\beta,I^n}(z_2) = x$ by (vii). Hence,

$$U^{-n}_\beta(x) \supseteq \bigl\{T^{-n}_{\beta,I^n}(x) : x \in T_{\beta,I^n}(\overline{I}^n),\, I^n \in W^n\bigr\}.$$

The second part of (ix) now follows. ☐

We will need the following standard lemma. Recall that $S_{n,I^n}\phi$ is defined in (3.19).

Lemma 3.25. *Fix $\beta > 1$. Suppose $\alpha \in (0,1]$ and $\phi \in C^{0,\alpha}(I)$. For all $n \in \mathbb{N}$, $I^n \in W^n$, and x, $y \in \overline{I}^n$, we have*

$$|S_{n,I^n}\phi(x) - S_{n,I^n}\phi(y)| \leqslant \frac{|\phi|_\alpha}{\beta^\alpha - 1} \bigl|T^n_{\beta,I^n}(x) - T^n_{\beta,I^n}(y)\bigr|^\alpha.$$

Proof. By (3.19), (3.17), and Proposition 3.24 (ii),

$$\begin{aligned}|S_{n,I^n}\phi(x)-S_{n,I^n}\phi(y)| &\leqslant |\phi|_\alpha\sum_{i=0}^{n-1}\bigl|T^i_{\beta,I^n}(x)-T^i_{\beta,I^n}(y)\bigr|^\alpha\\ &= |\phi|_\alpha\sum_{i=0}^{n-1}\bigl|T^n_{\beta,I^n}(x)-T^n_{\beta,I^n}(y)\bigr|^\alpha/\beta^{(n-i)\alpha}\\ &\leqslant \frac{|\phi|_\alpha}{\beta^\alpha-1}\bigl|T^n_{\beta,I^n}(x)-T^n_{\beta,I^n}(y)\bigr|^\alpha. \qquad\square\end{aligned}$$

4. Maximizing measures

In this section, we introduce the notion of *limit-maximizing measure*, which will be useful for a dynamical system, such as T_β, whose set of invariant measures is not necessarily weak* compact. For $\beta>1$ and $\phi\in C(I)$, we first show that the existence of a maximizing measure for (I,U_β,ϕ) is equivalent to the existence of a maximizing measure for $(X_\beta,\sigma,\phi\circ h_\beta)$. We then prove that a measure is limit-maximizing for (I,T_β,ϕ) if and only if it is maximizing for (I,U_β,ϕ).

Definition 4.1. Let $T\colon X\to X$ be a Borel measurable map on a compact metric space X. For a Borel measurable function $\psi\colon X\to\mathbb{R}$, a probability measure μ is called a (T,ψ)-*limit-maximizing measure*, or simply a ψ-*limit-maximizing measure*, if it is a weak* accumulation point of $\mathcal{M}(X,T)$ and $\int\psi\,\mathrm{d}\mu=Q(T,\psi)$. We denote the set of (T,ψ)-limit-maximizing measures by $\mathcal{M}^*_{\max}(T,\psi)$.

Clearly, $\mathcal{M}_{\max}(T,\psi)\subseteq\mathcal{M}^*_{\max}(T,\psi)$. For $\beta>1$, let us write

$$Z_\beta\coloneqq\{x\in I:\pi_\beta(x)\neq\pi^*_\beta(x)\}. \tag{4.1}$$

The following lemma collects together some basic properties of Z_β.

Lemma 4.2. *Given $\beta>1$, the following statements are true:*

(i) $Z_\beta=\bigl(\bigcup_{n\in\mathbb{N}}T_\beta^{-n}(0)\bigr)\smallsetminus\{0\}=\bigcup_{n\in\mathbb{N}}U_\beta^{-n}(1)$ *and in particular* $D_\beta\subseteq Z_\beta$.

(ii) $h_\beta^{-1}(W)=\pi^*_\beta(W)\cup\pi_\beta(W\cap Z_\beta)$ *for each* $W\subseteq I$ *and* $\pi_\beta(Z_\beta)\cap\pi^*_\beta(I)=\emptyset$.

(iii) *If* $x\in I\smallsetminus Z_\beta$, *then* $T^n_\beta(x)=U^n_\beta(x)$ *for all* $n\in\mathbb{N}$.

(iv) π_β *and* π^*_β *are continuous on* $I\smallsetminus Z_\beta$.

(v) $\mu(\pi_\beta(Z_\beta))=0$ *for all* $\mu\in\mathcal{M}(X_\beta,\sigma)$.

Proof. (i) Fix arbitrary $x\in I$. By (4.1), Proposition 3.9 (ii), and Lemma 3.8, $x\in Z_\beta$ if and only if there exists $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{N}$ with $z_n>0$ such that

$$\pi_\beta(x)=z_1\dots z_n(0)^\infty.$$

By Proposition 3.9 (iii) and (v), $T^n_\beta(x)=\bigl(T^n_\beta\circ h_\beta\circ\pi_\beta\bigr)(x)=(h_\beta\circ\sigma^n\circ\pi_\beta)(x)$. So the condition above is equivalent to the condition that there exists $n\in\mathbb{N}$ such that $x\in T_\beta^{-n}(0)\smallsetminus T_\beta^{-(n-1)}(0)$. Hence

$$Z_\beta=\bigcup_{n\in\mathbb{N}}\bigl(T_\beta^{-n}(0)\smallsetminus T_\beta^{-(n-1)}(0)\bigr)=\Bigl(\bigcup_{n\in\mathbb{N}}T_\beta^{-n}(0)\Bigr)\smallsetminus\{0\}.$$

Similarly, we can prove that $Z_\beta=\bigcup_{n\in\mathbb{N}}U_\beta^{-n}(1)$. In particular, by Proposition 3.13 (i), $D_\beta=U_\beta^{-1}(1)\subseteq Z_\beta$.

(ii) Fix arbitrary $W \subseteq I$. By Proposition 3.9 (xi), we have $h_\beta^{-1}(W) = \pi_\beta(W) \cup \pi_\beta^*(W)$. By (4.1),

$$\begin{aligned} h_\beta^{-1}(W) &= \pi_\beta^*(W) \cup \pi_\beta(W) = \pi_\beta^*(W) \cup (\pi_\beta(W \smallsetminus Z_\beta) \cup \pi_\beta(W \cap Z_\beta)) \\ &= \big(\pi_\beta^*(W) \cup \pi_\beta^*(W \smallsetminus Z_\beta)\big) \cup \pi_\beta(W \cap Z_\beta) = \pi_\beta^*(W) \cup \pi_\beta(W \cap Z_\beta). \end{aligned}$$

For each $A \in \pi_\beta(Z_\beta)$, A has finitely many nonzero terms and $(0)^\infty \notin \pi_\beta(Z_\beta)$ (see Proposition 3.9 (ii) and Lemma 3.8). By Proposition 3.9 (i), (ii), and Lemma 3.8, $\pi_\beta^*(x)$ has infinitely many nonzero terms for all $x \in (0,1]$ and $\pi_\beta^*(0) = (0)^\infty$. The second part of (ii) follows.

(iii) By Proposition 3.9 (iii) and (v), for each $x \in I$,

$$\begin{aligned} T_\beta^n(x) &= \big(T_\beta^n \circ h_\beta \circ \pi_\beta\big)(x) = (h_\beta \circ \sigma^n \circ \pi_\beta)(x) \quad \text{and} \\ U_\beta^n(x) &= \big(U_\beta^n \circ h_\beta \circ \pi_\beta^*\big)(x) = \big(h_\beta \circ \sigma^n \circ \pi_\beta^*\big)(x), \end{aligned}$$

so these, together with (4.1), give (iii).

(iv) Assume that $x \in (0,1] \smallsetminus Z_\beta$, so that $\pi_\beta(x) = \pi_\beta^*(x)$ by (4.1). From the fact that π_β^* is left-continuous, the fact that $\pi_\beta^*(y) \preceq \pi_\beta(y)$ for all $y \in I$, and the fact that π_β is strictly increasing (see Proposition 3.9 (ix), (ii), and (vi)), we have

$$\pi_\beta^*(x) = \lim_{y \nearrow x} \pi_\beta^*(y) \preceq \lim_{y \nearrow x} \pi_\beta(y) \preceq \pi_\beta(x).$$

So $\lim_{y \nearrow x} \pi_\beta(y) = \pi_\beta(x)$. Combining this with the fact that π_β is right-continuous on $[0,1)$, and that $x \in (0,1] \smallsetminus Z_\beta$ was arbitrary, we see that π_β is continuous on $I \smallsetminus Z_\beta$. The fact that π_β^* is continuous on $I \smallsetminus Z_\beta$ can be proved similarly.

(v) If $x \in Z_\beta$ then $(0)^\infty \in \mathcal{O}^\sigma(\pi_\beta(x))$ by Proposition 3.9 (ii) and Lemma 3.8. So

$$\pi_\beta(Z_\beta) \subseteq \left(\bigcup_{n=1}^{+\infty} \sigma^{-n}((0)^\infty) \right) \smallsetminus \{(0)^\infty\}. \tag{4.2}$$

If $\mu \in \mathcal{M}(X_\beta, \sigma)$ and $n \in \mathbb{N}$, then $\mu(\sigma^{-n}((0)^\infty)) = \mu(\{(0)^\infty\})$ and $(0)^\infty \in \sigma^{-n}((0)^\infty)$. This implies that

$$\mu\left(\left(\bigcup_{n=1}^{+\infty} \sigma^{-n}((0)^\infty) \right) \smallsetminus \{(0)^\infty\} \right) = 0,$$

and therefore $\mu(\pi_\beta(Z_\beta)) = 0$. □

Now we consider the relation between $\mathcal{M}(I, U_\beta)$ and $\mathcal{M}(X_\beta, \sigma)$.

Notation. Fix $\beta > 1$. Define

$$G_\beta \colon \mathcal{M}(I, U_\beta) \to \mathcal{M}(X_\beta, \sigma)$$

to be the pushforward of π_β^*, in other words,

$$G_\beta(\mu)(Y) := \mu\big(\big(\pi_\beta^*\big)^{-1}(Y)\big) \tag{4.3}$$

for each $\mu \in \mathcal{M}(I, U_\beta)$ and each Borel measurable subset $Y \subseteq X_\beta$. By Proposition 3.9 (iii), it is straightforward to check that G_β is well-defined. Define

$$H_\beta \colon \mathcal{M}(X_\beta, \sigma) \to \mathcal{P}(I)$$

by

$$H_\beta(\nu)(W) := \nu\big({h_\beta}^{-1}(W)\big) \tag{4.4}$$

for each $\nu \in \mathcal{M}(X_\beta, \sigma)$ and each Borel measurable subset $W \subseteq I$.

Proposition 4.3. *If $\beta > 1$ and $\phi \in C(I)$, then the following statements are true:*

(i) *H_β is a homeomorphism from $\mathcal{M}(X_\beta, \sigma)$ to $\mathcal{M}(I, U_\beta)$ with respect to the weak* topology, with $G_\beta^{-1} = H_\beta$.*

(ii) *$\mathcal{M}(I, U_\beta)$ is compact in the weak* topology.*

(iii) *$Q(U_\beta, \phi) = Q(\sigma|_{X_\beta}, \phi \circ h_\beta)$ and $\mathcal{M}_{\max}(U_\beta, \phi) = H_\beta(\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)) \neq \emptyset$.*

Proof. (i) By Proposition 3.9 (iii), $\sigma(\pi_\beta^*(I)) \subseteq \pi_\beta^*(I)$. Applying Lemma 4.2 (ii) to $W = I$, we have $X_\beta = \pi_\beta^*(I) \cup \pi_\beta(Z_\beta)$. Thus, by Lemma 4.2 (v), we have that $\mathcal{M}\bigl(\pi_\beta^*(I), \sigma\bigr)$ can be naturally identified with $\mathcal{M}(X_\beta, \sigma)$. More precisely, $\mu(\cdot) \mapsto \mu\bigl(\cdot \cap \pi_\beta^*(I)\bigr)$ is a homeomorphism from $\mathcal{M}\bigl(\pi_\beta^*(I), \sigma\bigr)$ to $\mathcal{M}(X_\beta, \sigma)$. So H_β can be seen as the pushforward of $h_\beta|_{\pi_\beta^*(I)}$ from $\mathcal{M}\bigl(\pi_\beta^*(I), \sigma\bigr)$ to $\mathcal{P}(I)$. Proposition 3.9 (v) implies that $H_\beta\bigl(\mathcal{M}(X_\beta, \sigma)\bigr) \subseteq \mathcal{M}(I, U_\beta)$.

For each $\mu \in \mathcal{M}(I, U_\beta)$ and each Borel measurable subset $Y \subseteq X_\beta$, by (4.3) and Proposition 3.9 (vi) and (iv), we have

$$G_\beta(\mu)(Y) = \mu\bigl(\bigl(\pi_\beta^*\bigr)^{-1}(Y)\bigr) = \mu\bigl(\bigl(\pi_\beta^*\bigr)^{-1}(Y \cap \pi_\beta^*(I))\bigr) = \mu\bigl(h_\beta\bigl(Y \cap \pi_\beta^*(I)\bigr)\bigr). \tag{4.5}$$

Hence we derive that $(H_\beta \circ G_\beta)(\mu) = \mu$ for all $\mu \in \mathcal{M}(I, U_\beta)$. More precisely, for each Borel measurable subset $W \subseteq I$, by (4.4), (4.5), Lemma 4.2 (ii), and Proposition 3.9 (v),

$$\begin{aligned}
(H_\beta \circ G_\beta)(\mu)(W) &= G_\beta(\mu)\bigl(h_\beta^{-1}(W)\bigr) \\
&= \mu\bigl(h_\beta\bigl(h_\beta^{-1}(W) \cap \pi_\beta^*(I)\bigr)\bigr) \\
&= \mu\bigl(h_\beta\bigl(\bigl(\pi_\beta^*(W) \cup \pi_\beta(W \cap Z_\beta)\bigr) \cap \pi_\beta^*(I)\bigr)\bigr) \\
&= \mu\bigl(h_\beta\bigl(\bigl(\pi_\beta^*(W) \cap \pi_\beta^*(I)\bigr) \cup \bigl(\pi_\beta(W \cap Z_\beta) \cap \pi_\beta^*(I)\bigr)\bigr)\bigr) \\
&= \mu\bigl(h_\beta\bigl(\pi_\beta^*(W)\bigr)\bigr) \\
&= \mu(W).
\end{aligned}$$

For each $\nu \in \mathcal{M}(X_\beta, \sigma)$ and each Borel measurable subset $W \subseteq I$, by (4.4) and Lemma 4.2 (ii) and (v), we have

$$H_\beta(\nu)(W) = \nu\bigl(h_\beta^{-1}(W)\bigr) = \nu\bigl(\pi_\beta^*(W) \cup \pi_\beta(W \cap Z_\beta)\bigr) = \nu\bigl(\pi_\beta^*(W)\bigr). \tag{4.6}$$

Hence we derive that $(G_\beta \circ H_\beta)(\nu) = \nu$ for all $\nu \in \mathcal{M}(X_\beta, \sigma)$. More precisely, for each Borel measurable subset $Y \subseteq X_\beta$, by (4.3) and (4.6),

$$(G_\beta \circ H_\beta)(\nu)(Y) = H_\beta(\nu)\bigl(\bigl(\pi_\beta^*\bigr)^{-1}(Y)\bigr) = \nu\bigl(\pi_\beta^*\bigl(\bigl(\pi_\beta^*\bigr)^{-1}(Y)\bigr)\bigr) = \nu(Y).$$

By the above, and Proposition 3.9 (x), H_β is a continuous bijection from $\mathcal{M}(X_\beta, \sigma)$ to $\mathcal{M}(I, U_\beta)$.

The weak* compactness of $\mathcal{M}(X_\beta, \sigma)$ follows immediately from the compactness of X_β and the continuity of σ. By [Wa82, Theorem 6.4], the set of probability measures on I is Hausdorff in the weak* topology, hence $\mathcal{M}(I, U_\beta)$ is Hausdorff, and therefore H_β is a homeomorphism from $\mathcal{M}(X_\beta, \sigma)$ to $\mathcal{M}(I, U_\beta)$, with $G_\beta^{-1} = H_\beta$.

(ii) follows immediately from (i), and the weak* compactness of $\mathcal{M}(X_\beta, \sigma)$.

(iii) Since G_β is the pushforward of π_β^*, for each $\mu \in \mathcal{M}(X_\beta, \sigma)$, by Proposition 3.9 (iv) and statement (i), we have

$$\int_I \phi \, \mathrm{d}H_\beta(\mu) = \int_I \bigl(\phi \circ h_\beta \circ \pi_\beta^*\bigr) \, \mathrm{d}H_\beta(\mu) = \int_{X_\beta} (\phi \circ h_\beta) \, \mathrm{d}\bigl(G_\beta \circ H_\beta\bigr)\mu = \int_{X_\beta} (\phi \circ h_\beta) \, \mathrm{d}\mu.$$

By (i), we obtain the required identities

$$Q(U_\beta, \phi) = Q(\sigma|_{X_\beta}, \phi \circ h_\beta) \text{ and } \mathcal{M}_{\max}\big(U_\beta, \phi\big) = H_\beta(\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)).$$

Since $\mathcal{M}(X_\beta, \sigma)$ is weak* compact and h_β is continuous (see Proposition 3.9 (x)), the set $\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)$ is non-empty. $\square$

Proposition 4.4. *If $\beta > 1$ and $\phi \in C(I)$, then the following statements are true:*

(i) *$\mathcal{M}(I, U_\beta)$ is equal to the weak* closure of $\mathcal{M}(I, T_\beta)$.*
(ii) *$Q(T_\beta, \phi) = Q(U_\beta, \phi)$.*
(iii) *$\mathcal{M}^*_{\max}(T_\beta, \phi) = \mathcal{M}_{\max}(U_\beta, \phi)$.*

Proof. If β is not a simple beta-number, then $\mathcal{M}(I, T_\beta) = \mathcal{M}(I, U_\beta)$ is weak* compact (see Proposition 3.13 (iv) and Proposition 4.3 (ii)). So (i) holds and both (ii) and (iii) follow immediately from (i).

If β is a simple beta-number, by Proposition 3.13 (v), it suffices to prove that $\mu_{\mathcal{O}'_\beta(1)}$ is contained in the weak* closure of $\mathcal{M}(I, T_\beta)$. Note that $\mathcal{O}'_\beta(1)$ and $\mathcal{O}^\sigma\big(\pi^*_\beta(1)\big)$ are periodic orbits of U_β and σ, respectively. By [Si76, p. 249], the periodic measures are weak* dense in $\mathcal{M}(X_\beta, \sigma)$, so there exists a sequence of periodic orbits $\{\mathcal{O}_n\}$ of (X_β, σ) satisfying:
(a) $\mathcal{O}_n \neq \mathcal{O}^\sigma(\pi^*_\beta(1))$ and $\mathcal{O}_n \neq \{(0)^\infty\}$ for all $n \in \mathbb{N}$, and
(b) $\mu_{\mathcal{O}_n}$ converges to $\mu_{\mathcal{O}^\sigma(\pi^*_\beta(1))}$ in the weak* topology as n tends to $+\infty$.

Since $\mathcal{O}_n$ is periodic and $(0)^\infty \notin \mathcal{O}_n$, by (4.2), we have $\mathcal{O}_n \cap \pi_\beta(Z_\beta) = \emptyset$. Recall that $X_\beta = \pi^*_\beta(I) \cup \pi_\beta(Z_\beta)$ (see Lemma 4.2 (ii)), so $\mathcal{O}_n \subseteq \pi^*_\beta(I)$.

By Proposition 3.9 (v), each $h_\beta(\mathcal{O}_n)$ is a periodic orbit of U_β and $h_\beta\big(\mathcal{O}^\sigma\big(\pi^*_\beta(1)\big)\big) = \mathcal{O}'_\beta(1)$. By Proposition 4.3 (i), $H_\beta(\mu_{\mathcal{O}_n}) = \mu_{h_\beta(\mathcal{O}_n)}$ then converges to $H_\beta\big(\mu_{\mathcal{O}^\sigma(\pi^*_\beta(1))}\big) = \mu_{\mathcal{O}'_\beta(1)}$ in the weak* topology, and $h_\beta(\mathcal{O}_n) \neq \mathcal{O}'_\beta(1)$ for each $n \in \mathbb{N}$. But $H_\beta(\mu_{\mathcal{O}_n}) \in \mathcal{M}(I, T_\beta)$ for each $n \in \mathbb{N}$, by Proposition 3.13 (iii), so (i) follows. Both (ii) and (iii) follow immediately from (i). $\square$

5. The Mañé lemma

The purpose of this section is to prove a version of the Mañé lemma for beta-transformations (Theorem 5.9), and derive a revelation theorem (Theorem 5.10), an important consequence regarding the support of a maximizing measure. A key tool is to introduce an operator analogous to the one used by Bousch [Bou00], and show (Proposition 5.7) that it has a fixed point function with certain regularity properties (following [GLT09], this fixed point can be referred to as a *calibrated sub-action*).

For a Borel measurable map $T \colon I \to I$, and bounded Borel measurable function $\psi \colon I \to \mathbb{R}$, to study the (T, ψ)-maximizing measures it is convenient, whenever possible, to consider a cohomologous function $\widetilde{\psi}$ satisfying $\widetilde{\psi} \leqslant Q(T, \psi)$. We recall the following (cf. [Je19, p. 2601]):

Definition 5.1. Suppose $T \colon I \to I$ is Borel measurable, and $\psi \colon I \to \mathbb{R}$ is bounded and Borel measurable. If $\psi \leqslant Q(T, \psi)$ and $\psi^{-1}(Q(T, \psi))$ contains $\operatorname{supp} \mu$ for some $\mu \in \mathcal{M}(I, T)$, then ψ is said to be *revealed*. If $Q(T, \psi) = 0$ then ψ is said to be *normalised*; in particular, a normalised function ψ is revealed if and only if $\psi \leqslant 0$ and $\psi^{-1}(0)$ contains $\operatorname{supp} \mu$ for some $\mu \in \mathcal{M}(I, T)$.

Lemma 5.2. *Suppose $T \colon I \to I$ is Borel measurable, $\phi \colon I \to \mathbb{R}$ is bounded and Borel measurable, and $\mathcal{M}_{\max}(T, \phi) \neq \emptyset$. Denote $\overline{\phi} = \phi - Q(T, \phi)$, and suppose $\widetilde{\phi} = \overline{\phi} + u - u \circ T$ for some bounded Borel measurable function $u \colon I \to \mathbb{R}$. Then the following statements are true:*

(i) $Q(T,\widetilde{\phi}) = Q(T,\overline{\phi}) = 0$.

(ii) $\mathcal{M}_{\max}(T,\,\phi) = \mathcal{M}_{\max}(T,\,\overline{\phi}) = \mathcal{M}_{\max}(T,\,\widetilde{\phi})$.

(iii) *If* $x \in I$ *is such that* $\widetilde{\phi} \leqslant 0$ *and* $\mathcal{O}^T(x) \subseteq \widetilde{\phi}^{-1}(0)$, *then* $\mathcal{O}^T(x)$ *is a* (T,ϕ)*-maximizing orbit.*

Proof. (i) and (ii) follow from (1.1), (1.2), and the fact that

$$\int \widetilde{\phi}\,\mathrm{d}\mu = \int (\overline{\phi} + u - u\circ T)\,\mathrm{d}\mu = \int \overline{\phi}\,\mathrm{d}\mu \text{ for all } \mu \in \mathcal{M}(I,T).$$

If $\widetilde{\phi} \leqslant 0$ and $\mathcal{O}^T(x) \subseteq \widetilde{\phi}^{-1}(0)$, then $0 = \frac{1}{n}S_n^T\widetilde{\phi}(x) = \frac{1}{n}S_n^T\overline{\phi}(x) + \frac{1}{n}(u(x) - u(T^n(x)))$ for all $n \in \mathbb{N}$, and (iii) follows from the fact that u is bounded. ☐

The following operator[6] $\mathcal{L}_\psi$ is an analogue of the one used by Bousch in [Bou00].

Definition 5.3. Let $\psi\colon I \to \mathbb{R}$ be bounded and Borel measurable. For $\beta > 1$, define $\mathcal{L}_\psi\colon \mathbb{R}^I \to \mathbb{R}^I$ by

$$\mathcal{L}_\psi(u)(x) \coloneqq \begin{cases} \max\limits_{y\in U_\beta^{-1}(x)} (u+\psi)(y) & \text{if } x \in (0,1], \\ \max\limits_{y\in T_\beta^{-1}(0)\smallsetminus\{1\}} (u+\psi)(y) & \text{if } x = 0. \end{cases} \tag{5.1}$$

Note that if $u\colon I \to \mathbb{R}$ is bounded then so is $\mathcal{L}_\psi(u)$. By (5.1) and Proposition 3.24 (ix), an equivalent definition of $\mathcal{L}_\psi$ is

$$\mathcal{L}_\psi(u)(x) \coloneqq \max\bigl\{(u+\psi)(y) : y = T_{\beta,I^1}^{-1}(x),\, I^1 \in W^1\bigr\}. \tag{5.2}$$

Lemma 5.4. *If* $\beta > 1$ *and* $\psi\colon I \to \mathbb{R}$ *is bounded and Borel measurable, and* $\overline{\psi} \coloneqq \psi - Q(T_\beta,\psi)$, *then the following statements are true:*

(i) *If* $x \in I$, $n \in \mathbb{N}$, *and* $u\colon I \to \mathbb{R}$ *is bounded, then*

$$\mathcal{L}_{\overline{\psi}}^n(u)(x) + nQ(T_\beta,\psi) = \mathcal{L}_\psi^n(u)(x) = \max\bigl\{u(y) + S_{n,I^n}\psi(y) : y = T_{\beta,I^n}^{-n}(x),\, I^n \in W^n\bigr\}.$$

(ii) *For all* $x \in (0,1]$ *and* $n \in \mathbb{N}$,

$$\mathcal{L}_\psi^n(\mathbb{0})(x) = \max\bigl\{S_{n,I^n}\psi(y) : y = T_{\beta,I^n}^{-n}(x),\, I^n \in W^n\bigr\} = \max\bigl\{S_n^{U_\beta}\psi(y) : y \in U_\beta^{-n}(x)\bigr\}.$$

(iii) *For all* $n \in \mathbb{N}$,

$$\mathcal{L}_\psi^n(\mathbb{0})(0) = \max\bigl\{S_{n,I^n}\psi(y) : y = T_{\beta,I^n}^{-n}(0),\, I^n \in W^n\bigr\} = \max\bigl\{S_n\psi(y) : y \in T_\beta^{-n}(0) \smallsetminus \{1\}\bigr\}.$$

(iv) $\mathcal{L}_\psi(\sup_{v\in\mathcal{A}} v) = \sup_{v\in\mathcal{A}} \mathcal{L}_\psi(v)$ *for any collection* $\mathcal{A}$ *of bounded real-valued functions on* I.

(v) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a pointwise convergent sequence of bounded real-valued functions on* I, *then* $\lim\limits_{n\to+\infty} \mathcal{L}_\psi(u_n) = \mathcal{L}_\psi\bigl(\lim\limits_{n\to+\infty} u_n\bigr)$, *where* $\lim\limits_{n\to+\infty}$ *denotes pointwise limit.*

Proof. (i) The first equality in (i) is immediate from (5.1) and the fact that $\overline{\psi} = \psi - Q(T_\beta,\psi)$, and the second is easily proved by (5.2), Proposition 3.24 (vi), and induction (cf. e.g. [JMU06, JMU07]).

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[6] Although non-linear, the operator $\mathcal{L}_\psi$ is *tropical linear* (see e.g. [LS24] for further development of this tropical functional analysis viewpoint; see also [BLL13]).

(ii) The first identity follows immediately from (i). For each $n \in \mathbb{N}$, each $x \in (0,1]$, and each $I^n \in W^n$, by Proposition 3.24 (ii) and (v), $T^{-n}_{\beta,I^n}(x)$ is not the left endpoint of I^n. So by Proposition 3.24 (vii) and (ix), we obtain

$$\begin{aligned}\max\bigl\{S_{n,I^n}\psi(y) : y = T^{-n}_{\beta,I^n}(x),\, I^n \in W^n\bigr\} &= \max\bigl\{S^{U_\beta}_n\psi(y) : y = T^{-n}_{\beta,I^n}(x),\, I^n \in W^n\bigr\}\\ &= \max\bigl\{S^{U_\beta}_n\psi(y) : y \in U^{-n}_\beta(x)\bigr\}.\end{aligned}$$

(iii) The first identity follows immediately from (i). For each $n \in \mathbb{N}$, each $x \in (0,1]$, and each $I^n \in W^n$, by Proposition 3.24 (ii) and (v), $T^{-n}_{\beta,I^n}(0)$ is the left endpoint of I^n. So by Proposition 3.24 (vii) and (ix), we obtain

$$\begin{aligned}\max\bigl\{S_{n,I^n}\psi(y) : y = T^{-n}_{\beta,I^n}(0),\, I^n \in W^n\bigr\} &= \max\bigl\{S_n\psi(y) : y = T^{-n}_{\beta,I^n}(0),\, I^n \in W^n\bigr\}\\ &= \max\bigl\{S_n\psi(y) : y \in T^{-n}_\beta(0) \smallsetminus \{1\}\bigr\}.\end{aligned}$$

(iv) follows readily from the fact that $\psi + \sup_{v\in\mathcal{A}} v = \sup_{v\in A}(\psi + v)$, and

$$\max_{y=T^{-1}_{\beta,I^1}(x),\, I^1\in W^1} \sup_{v\in\mathcal{A}}(\psi + v)(y) = \sup_{v\in\mathcal{A}} \max_{y=T^{-1}_{\beta,I^1}(x),\, I^1\in W^1}(\psi + v)(y).$$

(v) Define $v\colon I \to \mathbb{R}$ by $v(x) \coloneqq \lim_{n\to\infty} u_n(x)$ for all $x \in I$. Fix arbitrary $x \in I$ and $\epsilon > 0$. Then there exists $N = N(x,\epsilon) \in \mathbb{N}$ such that if $n \geqslant N$ then $|u_n(y) - v(y)| < \epsilon$ for each of the finitely many pre-images $y \in \bigl\{T^{-1}_{\beta,I^1}(x) : I^1 \in W^1\bigr\}$.

Fix $n \geqslant N$. Let y_1, $y_2 \in \bigl\{T^{-1}_{\beta,I^1}(x) : I^1 \in W^1\bigr\}$ satisfy $\mathcal{L}_\psi(u_n)(x) = (\psi + u_n)(y_1)$ and $\mathcal{L}_\psi(v)(x) = (\psi + v)(y_2)$, so that

$$\begin{aligned}&(\mathcal{L}_\psi(u_n) - \mathcal{L}_\psi(v))(x) \leqslant (\psi + u_n)(y_1) - (\psi + v)(y_1) = u_n(y_1) - v(y_1) < \epsilon \quad \text{and}\\ &(\mathcal{L}_\psi(u_n) - \mathcal{L}_\psi(v))(x) \geqslant (\psi + u_n)(y_2) - (\psi + v)(y_2) = u_n(y_2) - v(y_2) > -\epsilon.\end{aligned}$$

Then (v) follows. □

Notation. For $\beta > 1$ and $\alpha \in (0,1]$, we write

$$K_{\alpha,\beta} \coloneqq \frac{1}{\beta^\alpha - 1}.$$

Lemma 5.5. *Suppose $\beta > 1$, $\alpha \in (0,1]$, $\phi \in C^{0,\alpha}(I)$, and $n \in \mathbb{N}$. Then*

$$\mathcal{L}^n_\phi(u)(x) - \mathcal{L}^n_\phi(u)(y) \geqslant -K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha \tag{5.3}$$

for all $u \in C^{0,\alpha}(I)$, and all $x, y \in I$ with $x < y$.

If, moreover, for all $1 \leqslant i \leqslant n$ the interval $[x,y)$ does not contain $U^i_\beta(1)$, then

$$\bigl|\mathcal{L}^n_\phi(u)(x) - \mathcal{L}^n_\phi(u)(y)\bigr| \leqslant K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha. \tag{5.4}$$

Proof. Suppose $u \in C^{0,\alpha}(I)$ and x, $y \in I$ with $x < y$. By Lemma 5.4 (i), there exists $I^n \in W^n$ and $y' \in \overline{I}^n$ with $y' = T^{-n}_{\beta,I^n}(y)$ such that

$$\mathcal{L}^n_\phi(u)(y) = u(y') + S_{n,I^n}\phi(y'). \tag{5.5}$$

Since $x < y$, Proposition 3.24 (v) means that $x \in T^n_{\beta,I^n}\bigl(\overline{I}^n\bigr)$ as well, so $x' \coloneqq T^{-n}_{\beta,I^n}(x) \in \overline{I}^n$, and

$$\mathcal{L}^n_\phi(u)(x) \geqslant u(x') + S_{n,I^n}\phi(x'). \tag{5.6}$$

Combining (5.5) and (5.6) gives

$$\mathcal{L}^n_\phi(u)(x) - \mathcal{L}^n_\phi(u)(y) \geqslant S_{n,I^n}\phi(x') + u(x') - S_{n,I^n}\phi(y') - u(y'). \tag{5.7}$$

Since $x', y' \in \overline{I}^n$, Lemma 3.25 gives

$$S_{n,I^n}\phi(x') - S_{n,I^n}\phi(y') \geqslant -K_{\alpha,\beta}|\phi|_\alpha |x-y|^\alpha. \tag{5.8}$$

Now $u \in C^{0,\alpha}(I)$, so $u(x') - u(y') \geqslant -|u|_\alpha |x'-y'|^\alpha$ and $|x'-y'| = \beta^{-n}|x-y|$ by Proposition 3.24 (ii), so $u(x') - u(y') \geqslant -|u|_\alpha \beta^{-n\alpha}|x-y|^\alpha$. But $\beta^{-n\alpha} < K_{\alpha,\beta}$, so

$$u(x') - u(y') \geqslant -K_{\alpha,\beta}|u|_\alpha |x-y|^\alpha. \tag{5.9}$$

Combining (5.7), (5.8), and (5.9) gives the required inequality (5.3).

A similar argument can be used to establish the bound (5.4). Specifically, suppose that $x, y \in I$ and $n \in \mathbb{N}$ are such that $[x,y) \cap \left\{U_\beta(1), \dots, U_\beta^n(1)\right\} = \emptyset$, so that, by Proposition 3.24 (v), if $I^n \in W^n$ then $x \in T^n_{\beta,I^n}\left(\overline{I}^n\right)$ if and only if $y \in T^n_{\beta,I^n}\left(\overline{I}^n\right)$.

By Lemma 5.4 (i) there exists some $I^n \in W^n$ and $x'' \in \overline{I}^n$ with $x'' = T^{-n}_{\beta,I^n}(x)$, such that

$$\mathcal{L}^n_\phi(u)(x) = S_{n,I^n}\phi(x'') + u(x'').$$

Defining $y'' := T^{-n}_{\beta,I^n}(y)$, an argument analogous to the one above, using Lemma 3.25 and Lemma 5.4 (i), then gives

$$\begin{aligned}
\mathcal{L}^n_\phi(u)(x) - \mathcal{L}^n_\phi(u)(y) &\leqslant S_{n,I^n}\phi(x'') + u(x'') - S_{n,I^n}\phi(y'') - u(y'') \\
&\leqslant (\beta^\alpha - 1)^{-1}|\phi|_\alpha |x-y|^\alpha + |u|_\alpha |x''-y''|^\alpha \\
&\leqslant (\beta^\alpha - 1)^{-1}|\phi|_\alpha |x-y|^\alpha + |u|_\alpha \beta^{-n\alpha}|x-y|^\alpha \\
&\leqslant (\beta^\alpha - 1)^{-1}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha \\
&= K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha,
\end{aligned}$$

and (5.4) follows. ☐

Of particular interest in the following Corollary 5.6 will be the choice $u = \mathbb{0}$, the function that is identically zero on I, and evaluation of (5.3) at either endpoint of I:

Corollary 5.6. *Suppose $\beta > 1$ and $\alpha \in (0,1]$. If $\phi \in C^{0,\alpha}(I)$, $n \in \mathbb{N}$, and $x, y \in I$, then*

$$\mathcal{L}^n_\phi(\mathbb{0})(0) \geqslant \mathcal{L}^n_\phi(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha, \tag{5.10}$$

$$\mathcal{L}^n_\phi(\mathbb{0})(x) \geqslant \mathcal{L}^n_\phi(\mathbb{0})(1) - K_{\alpha,\beta}|\phi|_\alpha. \tag{5.11}$$

Proof. If $y = 0$ then (5.10) clearly holds. If $y > 0$ then (5.10) follows from (5.3) with $x := 0$ and $u := \mathbb{0}$, since in this case $|u|_\alpha = 0$ and $|x-y|^\alpha \leqslant 1$. Similarly, if $x = 1$ then (5.11) clearly holds, and if $x < 1$ then (5.11) follows from (5.3) with $y := 1$, $u := \mathbb{0}$. ☐

We are now able to find a fixed point u_ϕ of the operator $\mathcal{L}_{\overline{\phi}}$:

Proposition 5.7. *Suppose $\beta > 1$ and $\alpha \in (0,1]$. If $\phi \in C^{0,\alpha}(I)$ then the function $u_\phi \colon I \to \mathbb{R}$ given by*

$$u_\phi(x) := \limsup_{n \to +\infty} \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(x),\ x \in I, \tag{5.12}$$

where $\overline{\phi} := \phi - Q(T_\beta, \phi)$, satisfies the following properties:

(i) *u_ϕ is Borel measurable and $|u_\phi(x)| \leqslant 3K_{\alpha,\beta}|\phi|_\alpha$ for each $x \in I$.*

(ii) *If $a \in (0,1]$ then $\lim_{x \nearrow a} u_\phi(x)$ exists, and satisfies $\lim_{x \nearrow a} u_\phi(x) \geqslant u_\phi(a)$. If $a \in [0,1)$ then $\lim_{x \searrow a} u_\phi(x)$ exists, and satisfies $u_\phi(a) \geqslant \lim_{x \searrow a} u_\phi(x)$. In particular, if $a \in (0,1)$ then*

$$\lim_{x \nearrow a} u_\phi(x) \geqslant u_\phi(a) \geqslant \lim_{x \searrow a} u_\phi(x). \tag{5.13}$$

(iii) $|u_\phi(x) - u_\phi(y)| \leqslant K_{\alpha,\beta}|\phi|_\alpha |x-y|^\alpha$ *if* $0 \leqslant x < y \leqslant 1$ *satisfy* $[x,y) \cap \mathcal{O}'_\beta(1) = \emptyset$.

(iv) $\mathcal{L}_{\overline{\phi}}(u_\phi) = u_\phi$.

Proof. For each $n \in \mathbb{N}$ and $x \in I$, we write

$$p_n(x) \coloneqq \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(x) \qquad \text{and} \qquad q_n(x) \coloneqq \sup_{m \geqslant n} p_m(x). \tag{5.14}$$

Note that, for each $x \in I$, the sequence $\{q_n(x)\}_{n\in\mathbb{N}}$ is non-increasing and

$$u_\phi(x) = \lim_{n\to+\infty} q_n(x) = \limsup_{n\to+\infty} p_n(x).$$

(i) Fix $n \in \mathbb{N}$. By (5.4), p_n is continuous at all points except for $U_\beta(1), \dots, U^n_\beta(1)$, and hence Borel measurable. Combining this with (5.12), u_ϕ is Borel measurable. By Lemma 5.4 (iii), there exists an n-cylinder $I^n = I(a_1, a_2, \dots, a_n)$ such that

$$p_n(0) = \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(0) = S_{n,I^n}\overline{\phi}(y_n) = S_n\overline{\phi}(y_n) \tag{5.15}$$

where

$$y_n \coloneqq \frac{a_1}{\beta} + \cdots + \frac{a_n}{\beta^n} \in T^{-n}_\beta(0) \smallsetminus \{1\}. \tag{5.16}$$

Define

$$k \coloneqq \min\{i \in \mathbb{N}_0 : a_j = 0 \text{ for all } i+1 \leqslant j \leqslant n\}. \tag{5.17}$$

Case 1. If $k = 0$, we get that $y_n = 0$ and since 0 is a fixed point of T_β,

$$p_n(0) = n\overline{\phi}(0) \leqslant 0. \tag{5.18}$$

Case 2. If $k > 0$, y_n is the right endpoint of a k-full cylinder $I^k \coloneqq I(a_1, \dots, a_k - 1)$ by Proposition 3.24 (iii) and hence by Proposition 3.24 (iv) there is a T^k_{β,I^k}-fixed point z_n in $\overline{I}^k$. By (5.16), $T^k_{\beta,I^k}(z_n) = z_n \leqslant y_n < 1$. So by Proposition 3.24 (ii) and (v), we have $z_n \in I^k$ and $T^k_\beta(z_n) = T^k_{\beta,I^k}(z_n) = z_n$. Moreover,

$$S_{k,I^k}\overline{\phi}(z_n) = S_k\overline{\phi}(z_n) \leqslant kQ(T_\beta, \overline{\phi}) = 0. \tag{5.19}$$

Since 0 is a fixed point of T_β, $\overline{\phi}(0) \leqslant Q(T_\beta, \overline{\phi}) = 0$, and hence combining this with the fact that $T^k_\beta(y_n) = 0$ (see (5.16)), we obtain

$$S_n\overline{\phi}(y_n) = S_k\overline{\phi}(y_n) + S_{n-k}\overline{\phi}\big(T^k_\beta(y_n)\big) = S_k\overline{\phi}(y_n) + (n-k)\overline{\phi}(0) \leqslant S_k\overline{\phi}(y_n). \tag{5.20}$$

Since $T^i_\beta(y_n) \neq 0$ for each $0 \leqslant i \leqslant k-1$, we have $T^i_\beta(y_n) = U^i_\beta(y_n)$ for each $0 \leqslant i \leqslant k-1$ by Remark 3.3. Then by Proposition 3.24 (vii), we have

$$S_k\overline{\phi}(y_n) = S^{U_\beta}_k\overline{\phi}(y_n) = S_{k,I^k}\overline{\phi}(y_n). \tag{5.21}$$

Combining (5.15), (5.20), (5.21), Lemma 3.25, and (5.19) gives

$$p_n(0) = S_n\overline{\phi}(y_n) \leqslant S_k\overline{\phi}(y_n) = S_{k,I^k}\overline{\phi}(y_n) - S_{k,I^k}\overline{\phi}(z_n) + S_{k,I^k}\overline{\phi}(z_n) \leqslant K_{\alpha,\beta}|\phi|_\alpha. \tag{5.22}$$

Combining Corollary 5.6, (5.18), and (5.22) gives

$$p_n(x) \leqslant p_n(0) + K_{\alpha,\beta}|\phi|_\alpha \leqslant 2K_{\alpha,\beta}|\phi|_\alpha \quad \text{for all } x \in I, n \in \mathbb{N},$$

so from (5.12) we deduce the upper bound

$$u_\phi(x) \leqslant 2K_{\alpha,\beta}|\phi|_\alpha \quad \text{for all } x \in I. \tag{5.23}$$

We now seek to derive a lower bound on u_ϕ, via a lower bound on $p_n(x)$. Note that $Q(U_\beta, \overline{\phi}) = Q(T_\beta, \overline{\phi}) = 0$ (see Proposition 4.4 (ii)), therefore $Q\big(U_\beta, S^{U_\beta}_n\overline{\phi}\big) = 0$ for all $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. So by Proposition 4.3 (iii) there is a probability measure $\mu \in$

$\mathcal{M}_{\max}(U_\beta,\phi)$ such that $\int_I S_n^{U_\beta}\overline{\phi}\,d\mu = 0$, and combining this with the fact that $S_{n,I^n}\overline{\phi}$ is the continuous extension of $\bigl(S_n^{U_\beta}\overline{\phi}\bigr)\big|_{\widetilde{I}^n}$ (see Proposition 3.24 (vii)), we obtain

$$0 \leqslant \sup_{x\in I}\bigl\{S_n^{U_\beta}\overline{\phi}\bigr\} = \max\{S_{n,I^n}\overline{\phi}(w) : I^n \in W^n, w \in I^n\}.$$

So there exists $I^n \in W^n$ and $w_n \in I^n$ with

$$S_{n,I^n}\overline{\phi}(w_n) \geqslant 0. \tag{5.24}$$

Defining $y := T^n_{\beta,I^n}(w_n)$, Lemma 5.4 (i) and (5.24) give

$$\mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(y) = \max\bigl\{S_{n,I^n}\overline{\phi}(w) : w = T^{-n}_{\beta,I^n}(y),\ I^n \in W^n\bigr\} \geqslant S_{n,I^n}\overline{\phi}(w_n) \geqslant 0. \tag{5.25}$$

Combining (5.14), Corollary 5.6, and (5.25) gives

$$p_n(0) = \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(0) \geqslant \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha \geqslant -K_{\alpha,\beta}|\phi|_\alpha. \tag{5.26}$$

Let y_n and k be as in (5.16) and (5.17). When $k = 0$, we get $y_n = 0$ and $p_n(0) = n\overline{\phi}(0)$. Notice that $1/\beta^n$ is the right endpoint of $I^n = (0,\dots,0)$, combining this with (5.26) gives

$$p_n(1) \geqslant \sum_{i=1}^{n}\overline{\phi}\bigl(1/\beta^i\bigr) \geqslant p_n(0) - |\phi|_\alpha\sum_{i=1}^{n}\beta^{-i\alpha} \geqslant -2K_{\alpha,\beta}|\phi|_\alpha.$$

When $k > 0$, by Proposition 3.24 (iii) and (5.16), we have the full n-cylinder

$$J^n := I(0,\dots,0,a_1,a_2,\dots,a_k-1) \in W_0^n,$$

with the right endpoint equal to y_n/β^{n-k}, so since $a_k > 0$, $U_\beta\bigl(y_n/\beta^{n-k}\bigr) = 1$. Then by (5.14) and Lemma 5.4 (ii),

$$p_n(1) = \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(1) = \max\bigl\{S_n^{U_\beta}\overline{\phi}(z) : z \in U_\beta^{-n}(1)\bigr\} \geqslant S_n^{U_\beta}\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr). \tag{5.27}$$

Now by (5.21) and (3.2),

$$S_n^{U_\beta}\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) = S_k^{U_\beta}\overline{\phi}(y_n) + S_{n-k}^{U_\beta}\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) = S_k\overline{\phi}(y_n) + \sum_{i=1}^{n-k}\overline{\phi}\bigl(y_n/\beta^i\bigr). \tag{5.28}$$

Note that $\overline{\phi}\bigl(y_n/\beta^i\bigr) - \overline{\phi}(0) \geqslant -|\phi|_\alpha\bigl(y_n/\beta^i\bigr)^\alpha$ for $1 \leqslant i \leqslant n-k$, so

$$\sum_{i=1}^{n-k}\overline{\phi}\bigl(y_n/\beta^i\bigr) \geqslant (n-k)\overline{\phi}(0) - |\phi|_\alpha\sum_{i=1}^{n-k}\bigl(y_n/\beta^i\bigr)^\alpha. \tag{5.29}$$

Since $\sum_{i=1}^{n-k}\bigl(y_n/\beta^i\bigr)^\alpha \leqslant \sum_{i=1}^{\infty}\beta^{-i\alpha} = K_{\alpha,\beta}$, (5.29) gives

$$\sum_{i=1}^{n-k}\overline{\phi}\bigl(y_n/\beta^i\bigr) \geqslant (n-k)\overline{\phi}(0) - K_{\alpha,\beta}|\phi|_\alpha. \tag{5.30}$$

Combining (5.27), (5.28), (5.30) gives

$$p_n(1) \geqslant (n-k)\overline{\phi}(0) - K_{\alpha,\beta}|\phi|_\alpha + S_k\overline{\phi}(y_n). \tag{5.31}$$

However, (5.15) and (5.20) together give

$$(n-k)\overline{\phi}(0) + S_k\overline{\phi}(y_n) = p_n(0). \tag{5.32}$$

So combining (5.31), (5.32), and (5.26) gives

$$p_n(1) \geqslant p_n(0) - K_{\alpha,\beta}|\phi|_\alpha \geqslant -2K_{\alpha,\beta}|\phi|_\alpha. \tag{5.33}$$

If $x \in I$ then (5.14), Corollary 5.6, and (5.33) give

$$p_n(x) = \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(x) \geqslant \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(1) - K_{\alpha,\beta}|\phi|_\alpha = p_n(1) - K_{\alpha,\beta}|\phi|_\alpha \geqslant -3K_{\alpha,\beta}|\phi|_\alpha. \tag{5.34}$$

By (5.12) and (5.34),

$$u_\phi(x) \geqslant -3K_{\alpha,\beta}|\phi|_\alpha. \tag{5.35}$$

The bounds (5.23) and (5.35) together give the required inequality $|u_\phi(x)| \leqslant 3K_{\alpha,\beta}|\phi|_\alpha$, so (i) is proved.

(ii) If $x < y$ then taking $u = \mathbb{0}$ in Lemma 5.5 gives

$$\mathcal{L}^n_{\phi}(\mathbb{0})(x) \geqslant \mathcal{L}^n_{\phi}(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha,$$

and taking the limit supremum, together with (5.12), gives

$$u_\phi(x) \geqslant u_\phi(y) - K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha. \tag{5.36}$$

If $a \in [0,1)$ then in particular (5.36) holds for all $a < x < y$, so taking $\liminf_{x\searrow a}$ gives

$$\liminf_{x\searrow a} u_\phi(x) \geqslant u_\phi(y) - K_{\alpha,\beta}|\phi|_\alpha|a-y|^\alpha, \tag{5.37}$$

and taking $\limsup_{y\searrow a}$ in (5.37) gives

$$\liminf_{x\searrow a} u_\phi(x) \geqslant \limsup_{y\searrow a} u_\phi(y), \tag{5.38}$$

so $\lim_{x\searrow a} u_\phi(x)$ exists, as required. Now setting $x = a$ in (5.36), and taking $\lim_{y\searrow a} u_\phi(y)$, gives that

$$u_\phi(a) \geqslant \lim_{y\searrow a} u_\phi(y), \tag{5.39}$$

as required.

If $a \in (0,1]$, an analogous argument shows that $\lim_{x\nearrow a} u_\phi(x)$ exists, and moreover

$$\lim_{x\nearrow a} u_\phi(x) \geqslant u_\phi(a). \tag{5.40}$$

If $a \in (0,1)$, the required inequality (5.13) is immediate from (5.39) and (5.40).

(iii) Now suppose $x, y \in I$ with $x < y$ and $[x,y) \cap \mathcal{O}'_\beta(1) = \emptyset$. For any $\epsilon > 0$, by (5.14) and (5.12) there exists $N \in \mathbb{N}$ such that $|p_N(x) - u_\phi(x)| < \epsilon$ and $q_N(y) - u_\phi(y) < \epsilon$, so

$$u_\phi(x) - u_\phi(y) < p_N(x) - q_N(y) + 2\epsilon \leqslant p_N(x) - p_N(y) + 2\epsilon \leqslant K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha + 2\epsilon, \tag{5.41}$$

where the final inequality uses (5.4). Similarly, there exists $M \in \mathbb{N}$ such that $|p_M(y) - u_\phi(y)| < \epsilon$ and $q_M(x) - u_\phi(x) < \epsilon$, and an analogous calculation gives

$$u_\phi(x) - u_\phi(y) \geqslant -K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha - 2\epsilon. \tag{5.42}$$

Since $\epsilon > 0$ was arbitrary, (iii) follows from (5.41) and (5.42).

(iv) If $x \in I$ then by Lemma 5.4 (iv), (v), and (5.14),

$$\begin{aligned}\mathcal{L}_{\overline{\phi}}(u_\phi)(x) &= \mathcal{L}_{\overline{\phi}}\big(\lim_{n\to+\infty} q_n\big)(x) = \lim_{n\to+\infty} \mathcal{L}_{\overline{\phi}}\big(\sup_{m\geqslant n} \mathcal{L}^m_{\overline{\phi}}(\mathbb{0})(x)\big)\\ &= \lim_{n\to+\infty}\big(\sup_{m\geqslant n} \mathcal{L}^{m+1}_{\overline{\phi}}(\mathbb{0})(x)\big) = \lim_{n\to+\infty} q_{n+1}(x) = u_\phi(x). \qquad \square\end{aligned}$$

The following construction of the regularisations of the function u_ϕ is key to our revelation theorem (Theorem 5.10).

Definition 5.8. Fix $\beta > 1$ and $\alpha \in (0,1]$. For each $\phi \in C^{0,\alpha}(I)$, let u_ϕ be the function defined in (5.12). Since U_β is left-continuous and upper semi-continuous on $(0,1]$, we define a *sub-action* for (U_β, ϕ) by

$$u^-_{\beta,\phi}(x) := \begin{cases} u_\phi(0) & \text{if } x = 0, \\ \lim_{y \nearrow x} u_\phi(y) & \text{if } x \in (0,1]. \end{cases} \tag{5.43}$$

Since T_β is right-continuous and lower semi-continuous on $[0,1)$, we define a sub-action for (T_β, ϕ) by

$$u^+_{\beta,\phi}(x) := \begin{cases} \lim_{y \searrow x} u_\phi(y) & \text{if } x \in [0,1), \\ \lim_{y \searrow T_\beta(1)} u_\phi(y) - \overline{\phi}(1) & \text{if } x = 1. \end{cases} \tag{5.44}$$

We define the *left-continuous revealed version* $\widetilde{\phi}^-$ and the *right-continuous revealed version* $\widetilde{\phi}^+$ by

$$\widetilde{\phi}^- := \overline{\phi} + u^-_{\beta,\,\phi} - u^-_{\beta,\,\phi} \circ U_\beta, \tag{5.45}$$

$$\widetilde{\phi}^+ := \overline{\phi} + u^+_{\beta,\,\phi} - u^+_{\beta,\,\phi} \circ T_\beta. \tag{5.46}$$

We are now able to prove a Mañé lemma[7] for beta-transformations, involving the above sub-actions and revealed versions.

Theorem 5.9 (Mañé lemma for beta-transformations). *Suppose* $\beta > 1$, $\alpha \in (0,1]$, *and* $\phi \in C^{0,\alpha}(I)$. *Then the following statements are true:*

(i) $u^-_{\beta,\,\phi}$ *is bounded left-continuous and upper semi-continuous on* $(0,1]$, *and* $u^+_{\beta,\,\phi}$ *is bounded right-continuous and lower semi-continuous on* $[0,1)$.

(ii) $\widetilde{\phi}^- \leqslant 0$ *and* $\widetilde{\phi}^+ \leqslant 0$ *on* I. *The function* $\widetilde{\phi}^-$ *is left-continuous on* $(0,1]$, *and* $\widetilde{\phi}^+$ *is right-continuous on* $[0,1)$.

(iii) *If the closed interval* $[x,y] \subseteq I$ *is disjoint from the orbit* $\mathcal{O}'_\beta(1)$, *then*

$$\left| u^-_{\beta,\,\phi}(x) - u^-_{\beta,\,\phi}(y) \right| \leqslant K_{\alpha,\beta} |\phi|_\alpha |x-y|^\alpha \quad \text{and} \tag{5.47}$$

$$\left| u^+_{\beta,\,\phi}(x) - u^+_{\beta,\,\phi}(y) \right| \leqslant K_{\alpha,\beta} |\phi|_\alpha |x-y|^\alpha. \tag{5.48}$$

Remark. If β is a beta-number, then $\mathcal{O}'_\beta(1)$ is a finite set, and by Theorem 5.9 (iii), the functions $u^-_{\beta,\,\phi}$ and $u^+_{\beta,\,\phi}$ are piecewise α-Hölder.

Proof. (i) follows immediately from (5.43), (5.44), and Proposition 5.7 (ii).

(ii) Since $T_\beta(0) = U_\beta(0) = 0$ (see (3.1) and (3.2)), by (5.45) and (5.46),

$$\widetilde{\phi}^-(0) = \widetilde{\phi}^+(0) = \overline{\phi}(0) \leqslant 0. \tag{5.49}$$

By (5.1) and Proposition 5.7 (iv),

$$u_{\beta,\phi}(x) = \max\bigl\{ \overline{\phi}(y) + u_{\beta,\phi}(y) : y \in U_\beta^{-1}(x) \bigr\} \quad \text{for } x \in (0,1] \text{ and} \tag{5.50}$$

$$u_{\beta,\phi}(0) = \max\bigl\{ \overline{\phi}(y) + u_{\beta,\phi}(y) : y \in T_\beta^{-1}(0) \smallsetminus \{1\} \bigr\}. \tag{5.51}$$

So for all $x \in (0,1]$, since $U_\beta(x) \in (0,1]$ (see (3.2)), then by (5.50) we get

$$\overline{\phi}(x) + u_{\beta,\phi}(x) \leqslant u_{\beta,\phi}(U_\beta(x)). \tag{5.52}$$

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[7] The term *Mañé lemma* originated in [Bou00], in view of the resemblance to a result of Mañé [Ma96, Theorem B] in the context of Lagrangian flows.

Combining this inequality with (5.45), (5.43), and Lemma 3.4 (i), for all $x \in (0,1]$, we obtain

$$\widetilde{\phi}^-(x) = \overline{\phi}(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(U_\beta(x)) \leqslant 0. \tag{5.53}$$

Combining (5.49) and (5.53) gives $\widetilde{\phi}^- \leqslant 0$ on I. By (5.46) and (5.44),

$$\widetilde{\phi}^+(1) = \overline{\phi}(1) + u^+_{\beta,\phi}(1) - u^+_{\beta,\phi}(T_\beta(1)) = 0. \tag{5.54}$$

Fix $x \in (0,1)$. If $T_\beta(x) \neq 0$, then $T_\beta(x) = U_\beta(x)$ (see Remark 3.3) and by (5.52),

$$\overline{\phi}(x) + u_{\beta,\phi}(x) \leqslant u_{\beta,\phi}(U_\beta(x)) = u_{\beta,\phi}(T_\beta(x)).$$

If $T_\beta(x) = 0$, then by (5.51),

$$\overline{\phi}(x) + u_{\beta,\phi}(x) \leqslant u_{\beta,\phi}(0) = u_{\beta,\phi}(T_\beta(x)).$$

Combining the two inequalities above with (5.46), (5.44), and Lemma 3.4 (i), for all $x \in (0,1)$, we obtain

$$\widetilde{\phi}^+(x) = \overline{\phi}(x) + u^+_{\beta,\phi}(x) - u^+_{\beta,\phi}(T_\beta(x)) \leqslant 0. \tag{5.55}$$

Combining (5.49), (5.54), and (5.55) gives $\widetilde{\phi}^+ \leqslant 0$ on I. The second part of (ii) follows from (i) and Lemma 3.4 (i).

(iii) This follows immediately from (5.43), (5.44), and Proposition 5.7 (iii). □

The importance of Theorem 5.9 is in allowing us to establish a form of *revelation theorem* (cf. [Je19, Section 5]): the following Theorem 5.10 localises the support of a maximizing measure as lying in the union of the zero sets of the revealed versions $\widetilde{\phi}^-$ and $\widetilde{\phi}^+$, and reveals that individual points in the support of such a measure have their full orbit, under either U_β or T_β, contained in either $\big(\widetilde{\phi}^+\big)^{-1}(0)$ or $\big(\widetilde{\phi}^-\big)^{-1}(0)$ respectively. This latter fact will be exploited in our proof of Theorem 1.2, more specifically in establishing the key Lemma 7.14.

Theorem 5.10 (Revelation theorem). *Suppose $\beta > 1$, $\alpha \in (0,1]$, $\phi \in C^{0,\alpha}(I)$, and $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$. Then the following statements are true:*

- (i) *If $x \in \operatorname{supp}\mu$, then either $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}'_\beta(x)$, or $\widetilde{\phi}^+ \equiv 0$ on $\mathcal{O}_\beta(x)$.*
- (ii) *If $1 \in \operatorname{supp}\mu$ then $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}'_\beta(1)$.*

In particular, $\operatorname{supp}\mu \subseteq \big(\widetilde{\phi}^+\big)^{-1}(0) \cup \big(\widetilde{\phi}^-\big)^{-1}(0)$.

Proof. (i) Since $\widetilde{\phi}^- = \overline{\phi} + u^-_{\beta,\phi} - u^-_{\beta,\phi} \circ U_\beta$ (see (5.45)) where $u^-_{\beta,\phi}$ is bounded and Borel measurable (see Proposition 5.7 (i) and (5.43)), and $\overline{\phi} = \phi - Q(U_\beta, \phi)$ is normalised, it follows from Lemma 5.2 (i) and (ii) that $Q(U_\beta, \widetilde{\phi}^-) = Q(U_\beta, \overline{\phi}) = 0$ and $\mathcal{M}_{\max}(U_\beta, \phi) = \mathcal{M}_{\max}(U_\beta, \overline{\phi}) = \mathcal{M}_{\max}(U_\beta, \widetilde{\phi}^-)$. So

$$\int \widetilde{\phi}^- \,\mathrm{d}\mu = 0. \tag{5.56}$$

Suppose $x \in \operatorname{supp}\mu$, so that $\mu((x-\epsilon, x+\epsilon)) > 0$ for all $\epsilon > 0$. Therefore, either $\mu((x-\epsilon, x]) > 0$ for all $\epsilon > 0$, or $\mu((x, x+\epsilon)) > 0$ for all $\epsilon > 0$.

First suppose that

$$\mu((x-\epsilon, x]) > 0 \text{ for all } \epsilon > 0, \tag{5.57}$$

and in this case we aim to show that

$$\widetilde{\phi}^- \equiv 0 \text{ on } \mathcal{O}'_\beta(x). \tag{5.58}$$

If $x = 0$, then (5.57) means that $\mu(\{0\}) > 0$, and since $\widetilde{\phi}^- \leqslant 0$, it follows that

$$0 = \int \widetilde{\phi}^- \,\mathrm{d}\mu \leqslant \mu(\{0\})\widetilde{\phi}^-(0),$$

so that $\widetilde{\phi}^-(0) = 0$, in other words $\widetilde{\phi}^- \equiv 0$ on $\{0\} = \mathcal{O}'_\beta(0)$, so (5.58) holds.

If $x \neq 0$, we first claim that

$$\mu((y-\epsilon, y]) > 0 \text{ for all } y \in \mathcal{O}'_\beta(x) \text{ and } \epsilon > 0. \tag{5.59}$$

To prove (5.59), note that $0 \notin \mathcal{O}'_\beta(x)$ since $x \neq 0$ (see Proposition 3.13 (i)). Assume that $y \in \mathcal{O}'_\beta(x)$ then $y = U^k_\beta(x)$ for some $k \in \mathbb{N}$. By Lemma 3.4 (i), for all $\epsilon > 0$ there exists $\delta > 0$ such that $U^k_\beta((x-\delta, x]) \subseteq (y-\epsilon, y]$. But $\mu \in \mathcal{M}(I, U_\beta)$, so (5.57) implies that $\mu((y-\epsilon, y]) = \mu\big(U_\beta^{-k}(y-\epsilon, y]\big) \geqslant \mu((x-\delta, x]) > 0$. So (5.59) holds.

Now $\widetilde{\phi}^-$ is left-continuous by Theorem 5.9 (ii), so if $y \in \mathcal{O}'_\beta(x)$ were such that $\widetilde{\phi}^-(y) < 0$, then there would exist $\rho, \epsilon > 0$ with $\widetilde{\phi}^-|_{(y-\epsilon,y]} < -\rho$, and (5.59) would imply that

$$\int \widetilde{\phi}^- \,\mathrm{d}\mu \leqslant \int_{(y-\epsilon,y]} \widetilde{\phi}^- \,\mathrm{d}\mu < -\rho\, \mu((y-\epsilon, y]) < 0,$$

which contradicts (5.56). So $\widetilde{\phi}^-(y)$ cannot be strictly negative, but on the other hand $\widetilde{\phi}^- \leqslant 0$ by Theorem 5.9 (ii), therefore $\widetilde{\phi}^-(y) = 0$. Since y was an arbitrary point in $\mathcal{O}'_\beta(x)$, we have shown that $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}'_\beta(x)$, which is the desired statement (5.58).

Now suppose that

$$\mu((x, x+\epsilon)) > 0 \text{ for all } \epsilon > 0, \tag{5.60}$$

and in this case we aim to show that

$$\widetilde{\phi}^+ \equiv 0 \text{ on } \mathcal{O}_\beta(x).$$

Note that (5.60) implies that $x \neq 1$, and consequently $1 \notin \mathcal{O}_\beta(x)$ by Proposition 3.13 (i). By arguments analogous to the ones above, Lemma 3.4 (i) first implies that $\mu((y, y+\epsilon)) > 0$ for all $y \in \mathcal{O}_\beta(x)$ and $\epsilon > 0$, and then the right-continuity of u^+_ϕ, and hence of $\widetilde{\phi}^+$, implies that $\widetilde{\phi}^+(y) = 0$ for each $y \in \mathcal{O}_\beta(x)$, as required.

(ii) If $1 \in \operatorname{supp}\mu$, then (5.57) holds for $x = 1$, and the argument in (i) above gives (5.58), in other words $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}'_\beta(1)$, as required. □

Remark 5.11. A consequence of Theorem 5.9, and a counterpoint to Theorem 5.10, is that ϕ-maximizing measures can be characterised in terms of the support of their pushforward under π^*_β: specifically, for all $\beta > 1$, $\alpha \in (0,1]$, and $\phi \in C^{0,\alpha}(I)$, there exists a closed subset $K \subseteq X_\beta$ such that a U_β-invariant measure μ belongs to $\mathcal{M}_{\max}(U_\beta, \phi)$ if and only if $\operatorname{supp} G_\beta(\mu) \subseteq K$. To see this, recall from Proposition 4.3 that $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ if and only if $G_\beta(\mu) \in \mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)$, and then the continuity of $\sigma|_{X_\beta}$ and $\phi \circ h_\beta$ means that the existence of such a K (a so-called *maximizing set* in the terminology of Morris [Mo07]) is guaranteed (using [Mo07, Theorem 1, Proposition 1]) if

$$\sup_{n\in\mathbb{N}} \sup_{A\in X_\beta} S^\sigma_n(\overline{\phi} \circ h_\beta)(A) < +\infty. \tag{5.61}$$

The bound (5.61) can be proved using $\widetilde{\phi}^-$ and $\widetilde{\phi}^+$, since the sub-actions $u^-_{\beta,\phi}$, $u^+_{\beta,\phi}$ are bounded, together with the fact that $X_\beta = \pi_\beta(I) \cup \pi^*_\beta(I)$ implied by Lemma 4.2 (ii). More precisely, if

$A \in \pi_\beta^*(I)$, with $A = \pi_\beta^*(x)$, then parts (iv) and (v) of Proposition 3.9, together with (5.45), give

$$S_n^\sigma(\overline{\phi} \circ h_\beta)(A) = S_n^{U_\beta}\overline{\phi}(x) = S_n^{U_\beta}\widetilde{\phi}^-(x) + u_\phi^-\bigl(U_\beta^n(x)\bigr) - u_\phi^-(x) \quad \text{for all } n \in \mathbb{N},$$

and using that $\widetilde{\phi}^- \leqslant 0$ (by Theorem 5.9), together with the bound on the modulus of u_ϕ from Proposition 5.7 (i) and the definition of u_ϕ^-, yields

$$S_n^\sigma(\overline{\phi} \circ h_\beta)(A) \leqslant 6K_{\alpha,\beta}|\phi|_\alpha \quad \text{for all } n \in \mathbb{N}. \tag{5.62}$$

If $A \in \pi_\beta(I)$ then a similar argument, using $\widetilde{\phi}^+$, gives

$$S_n^\sigma(\overline{\phi} \circ h_\beta)(A) \leqslant 2(3K_{\alpha,\beta} + 1)|\phi|_\alpha \quad \text{for all } n \in \mathbb{N} \tag{5.63}$$

(the righthand side of (5.63) differing from that of (5.62) due to the way u_ϕ^+ is defined at the point 1), and (5.62), (5.63) together imply (5.61).

Remark 5.12. In view of Remark 5.11, that characterises maximizing measures in terms of some support being contained in a maximizing set, a natural question is whether the condition that $\operatorname{supp}\mu \subseteq \bigl(\widetilde{\phi}^+\bigr)^{-1}(0) \cup \bigl(\widetilde{\phi}^-\bigr)^{-1}(0)$, which by Theorem 5.10 is necessary for a U_β-invariant measure to be ϕ-maximizing, is actually *sufficient*. The answer is no, as we explain below; note that this contrasts with more familiar systems (e.g. open distance-expanding maps) where a Mañé lemma can be shown to hold, and the sub-action (and hence the revealed version) can be shown to be continuous.

Let $\beta \approx 2.48119$ be the largest root of the cubic polynomial $\zeta^3 - 2\zeta^2 - 2\zeta + 2$. This is a non-simple beta-number, with $\pi_\beta(1) = 2(10)^\infty$.

Define the fixed point $z := h_\beta\left((1)^\infty\right) = \frac{1}{\beta-1} \approx 0.675$, and the two period-2 points $x := h_\beta\left((10)^\infty\right) = \frac{\beta}{\beta^2-1} \approx 0.481$, $y := h_\beta\left((01)^\infty\right) = \frac{1}{\beta^2-1} \approx 0.194$. We will exhibit Lipschitz functions ϕ such that the periodic measure $\mu := \frac{1}{2}(\delta_x + \delta_y)$ is not (U_β, ϕ)-maximizing, yet its support $\{x, y\}$ is contained in $\bigl(\widetilde{\phi}^+\bigr)^{-1}(0) \cup \bigl(\widetilde{\phi}^-\bigr)^{-1}(0)$.

Defining $\tau(s) := (s+1)/\beta$, let $x_1 := h_\beta\left(1(10)^\infty\right) = \tau(x) = \frac{\beta^2+\beta-1}{\beta^3-\beta} \approx 0.597$, and define sequences

$$t_i := h_\beta\left((1)^i 2(10)^\infty\right) = h_\beta\left((1)^i \pi_\beta(1)\right) = \tau^i(1) = \beta^{-i}\left(2 + \sum_{j=1}^{i-1}\beta^j\right), \quad i \in \mathbb{N},$$

$$y_i := h_\beta\left((1)^{i-1}0(01)^\infty\right) = \tau^{i-1}(y/\beta) = \frac{\beta^{i+1} + \beta^i - \beta^2 - \beta + 1}{\beta^i(\beta^2 - 1)}, \quad i \in \mathbb{N}.$$

Note in particular that $\{t_i\}_{i\in\mathbb{N}}$ is a backwards orbit of 1 (under U_β, but not T_β) converging to z, that $\{y_i\}_{i\in\mathbb{N}}$ is a backwards orbit of y (under both T_β and U_β), also converging to z, that $T_\beta^{-1}(x) = U_\beta^{-1}(x) = \{y, x_1, 1\}$, and that the relative ordering of the various points is

$$0 < y_1 < y < y_2 < x < y_3 < x_1 < y_4 < y_5 < y_6 < \cdots < z < \cdots < t_3 < t_2 < t_1 < 1.$$

For $\alpha \in (0,1]$, let $\phi \in C^{0,\alpha}(I)$ be non-positive, with $\phi(y) = -1$ and $\phi(x_1) = -2$, and identically zero on the points x, 1, and all points in the two sequences $\{t_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$. (For concreteness, ϕ might be chosen to be the function that is identically zero on the intervals $[0, y_1]$, $[y_2, y_3]$, and $[y_4, 1]$, and affine on each of $[y_1, y]$, $[y, y_2]$, $[y_3, x_1]$, $[x_1, y_4]$, with $\phi(y) = -1$ and $\phi(x_1) = -2$, though our analysis does not assume this). Note that ϕ is normalised, i.e., $Q(U_\beta, \phi) = 0$, since it attains its maximum value 0 at the fixed point z. On the other hand, $\int \phi \,\mathrm{d}\mu = -1/2 < Q(U_\beta, \phi)$, so $\mu \notin \mathcal{M}_{\max}(U_\beta, \phi)$.

For this value of β (and indeed whenever β is a beta-number), it can be shown that the sub-actions $u^-_{\beta,\phi}$, $u^+_{\beta,\phi}$, defined in (5.43), (5.44), satisfy $u^-_{\beta,\phi}(r) = \limsup_{n\to+\infty} \max\bigl\{S_n^{U_\beta}\phi(s) : s \in U_\beta^{-n}(r)\bigr\}$ for all $r \in (0,1]$, and $u^+_{\beta,\phi}(r) = \limsup_{n\to+\infty} \max\bigl\{S_n^{T_\beta}\phi(s) : s \in T_\beta^{-n}(r)\bigr\}$ for all $r \in [0,1)$. Now for each $n \in \mathbb{N}$, the point $y_n \in U_\beta^{-n}(y) \cap T_\beta^{-n}(y)$, and $S_n^{U_\beta}\phi(y_n) = 0 = S_n^{T_\beta}\phi(y_n)$, thus $u^-_{\beta,\phi}(y) = 0 = u^+_{\beta,\phi}(y)$. For each $n \geqslant 2$, the point $t_{n-1} \in U_\beta^{-n}(x)$, and $S_n^{U_\beta}\phi(t_{n-1}) = 0$, thus $u^+_{\beta,\phi}(x) = 0$. Since the point 1 does not have any pre-images under T_β, the points t_{n-1} do not belong to $T_\beta^{-n}(x)$. Apart from 1, the other two pre-images of x are y and x_1, and the inequality $\phi(y) > \phi(x_1)$, together with the fact that $\phi \equiv 0$ on the backwards orbit $\{y_n\}_{n\in\mathbb{N}}$ of y, means that $\max\bigl\{S_n^{T_\beta}\phi(s) : s \in T_\beta^{-n}(x)\bigr\} = \phi(y) = -1$ for all $n \in \mathbb{N}$, and hence $u^+_{\beta,\phi}(x) = -1$.

Finally, we can evaluate the left-continuous and right-continuous revealed versions at, respectively, the points x and y, to find (cf. (5.45) and (5.46)) that

$$\widetilde{\phi}^-(x) = \overline{\phi}(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(U_\beta(x)) = \phi(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(y) = 0 + 0 + 0 = 0$$

and

$$\widetilde{\phi}^+(y) = \overline{\phi}(y) + u^+_{\beta,\phi}(y) - u^+_{\beta,\phi}(T_\beta(y)) = \phi(y) + u^+_{\beta,\phi}(y) - u^+_{\beta,\phi}(x) = -1 + 0 + 1 = 0,$$

so $\operatorname{supp}\mu = \{x, y\} \subseteq \bigl(\widetilde{\phi}^+\bigr)^{-1}(0) \cup \bigl(\widetilde{\phi}^-\bigr)^{-1}(0)$, as claimed.

6. Emergent numbers

In this section we define the notion of emergent number, establish several alternative characterisations, and show that emergent numbers constitute a small subset of $(1, +\infty)$.

Definition 6.1. A real number $\beta > 1$ will be called *emergent* if

$$\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))} \cap \mathcal{S}_\gamma = \emptyset \text{ for all } \gamma \in (1, \beta).$$

The set of emergent numbers will be denoted by $\mathcal{E}$, and called the *emergent set*.

Remark 6.2. The intuition behind the terminology *emergent*, as described in Section 1, and in view of Lemma 3.17, is that the particular symbolic dynamics of the upper beta-expansion $\pi^*_\beta(1)$ has newly *emerged* at the value β, in the sense that it does not resemble any that has been witnessed for $\gamma \in (1, \beta)$. Note in particular that simple beta-numbers are emergent, whereas non-simple beta-numbers are not emergent.

Recalling from (4.1) that $Z_\beta = \{x \in I : \pi_\beta(x) \neq \pi^*_\beta(x)\}$, the following proposition gives two alternative characterisations of emergent numbers.

Proposition 6.3. *Consider $\beta > 1$. Then the following are equivalent:*

(i) *β is emergent.*

(ii) *$\overline{\mathcal{O}'_\beta(1)} \cap H^\gamma_\beta \subseteq Z_\beta$ for each $\gamma \in (1, \beta)$.*

(iii) *$\bigl(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}, \sigma\bigr)$ is minimal.*

Proof. (i) implies (ii): Assume that β is emergent. Let us suppose, for a contradiction, that the result is false, i.e., that there exists $\gamma \in (1, \beta)$ and $x \in I$ satisfying

$$x \in \bigl(\overline{\mathcal{O}'_\beta(1)} \cap H^\gamma_\beta\bigr) \smallsetminus Z_\beta.$$

By Proposition 3.9 (iii), we have $\mathcal{O}^\sigma\bigl(\pi^*_\beta(1)\bigr) = \pi^*_\beta\bigl(\mathcal{O}'_\beta(1)\bigr)$. Since π^*_β is continuous at x (see Lemma 4.2 (iv)), we have $\pi^*_\beta(x) \in \overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}$. Since $\pi_\beta|_{H^\gamma_\beta}$ is a homeomorphism (see

Lemma 3.21) and $x \in H_\beta^\gamma$, we have $\pi_\beta(x) \in \mathcal{S}_\gamma$. Hence, since $x \notin Z_\beta$ we have $\pi_\beta(x) = \pi_\beta^*(x) \in \overline{\mathcal{O}^\sigma(\pi_\beta^*(1))} \cap \mathcal{S}_\gamma$, which contradicts (see Definition 6.1) the fact that β is emergent.

(ii) implies (i): Assume that β is non-emergent. Then there exists $\gamma \in (1,\beta)$ such that $\mathcal{K}_\gamma := \mathcal{S}_\gamma \cap \overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}$ is a non-empty closed subset of $\mathcal{S}_\gamma$ with $\sigma(\mathcal{K}_\gamma) \subseteq \mathcal{K}_\gamma$, by Definition 6.1. Then by Lemma 3.21 and Proposition 3.9 (v), $h_\beta(\mathcal{K}_\gamma)$ is a non-empty closed subset of H_β^γ with $T_\beta(h_\beta(\mathcal{K}_\gamma)) \subseteq h_\beta(\mathcal{K}_\gamma)$. Since h_β is continuous and $\mathcal{O}'_\beta(1) = h_\beta\big(\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)\big)$ (see Proposition 3.9 (x), (iii), and (v)), we obtain $h_\beta(\mathcal{K}_\gamma) \subseteq \overline{\mathcal{O}'_\beta(1)}$. Hence $h_\beta(\mathcal{K}_\gamma) \subseteq \overline{\mathcal{O}'_\beta(1)} \cap H_\beta^\gamma$.

If $h_\beta(\mathcal{K}_\gamma) \subseteq Z_\beta$, then from the fact that $T_\beta(h_\beta(\mathcal{K}_\gamma)) \subseteq h_\beta(\mathcal{K}_\gamma)$, and Lemma 4.2 (i), we have both that $0 \in h_\beta(\mathcal{K}_\gamma)$ and $0 \notin Z_\beta$, which is a contradiction. If on the other hand $h_\beta(\mathcal{K}_\gamma) \nsubseteq Z_\beta$ then there exists $x \in h_\beta(\mathcal{K}_\gamma) \smallsetminus Z_\beta$. In both cases, $\big(\overline{\mathcal{O}'_\beta(1)} \cap H_\beta^\gamma\big) \smallsetminus Z_\beta$ is non-empty.

(i) implies (iii): Assume that β is emergent. Fix an arbitrary $A \in \overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}$. If $\pi_\beta^*(1) \in \overline{\mathcal{O}^\sigma(A)}$, then $\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))} = \overline{\mathcal{O}^\sigma(A)}$. So to show that $\big(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\big)$ is minimal, it suffices to verify that $\pi_\beta^*(1) \in \overline{\mathcal{O}^\sigma(A)}$. If on the contrary $\pi_\beta^*(1) \notin \overline{\mathcal{O}^\sigma(A)}$, then $\overline{\mathcal{O}^\sigma(A)}$ is a closed subset of $\mathcal{S}_\beta$ disjoint from $\pi_\beta^*(1)$, which is the maximal element in $\mathcal{S}_\beta$ (see (3.8) and (3.11)). Hence, from the fact that $\lim_{\gamma \nearrow \beta} \pi_\gamma^*(1) = \pi_\beta^*(1)$ (see Proposition 3.9 (xiv) and (3.10)), there exists $\gamma \in (1,\beta)$ with $\sigma^n(A) \preceq \pi_\gamma^*(1)$ for all $n \in \mathbb{N}_0$. Hence by (3.11), $A \in \overline{\mathcal{O}^\sigma(\pi_\beta^*(1))} \cap \mathcal{S}_\gamma$, which contradicts the assumption that β is emergent.

(iii) implies (i): Assume that β is non-emergent. By Definition 6.1 there exists $\gamma \in (1,\beta)$ and some $B \in \overline{\mathcal{O}^\sigma(\pi_\beta^*(1))} \cap \mathcal{S}_\gamma$. Hence the closure of $\mathcal{O}^\sigma(B)$ is contained in $\mathcal{S}_\gamma$, and therefore $\big(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\big)$ is not minimal. □

Example 6.4. As a specific example of an emergent number that is not a simple beta-number, let $F_0 = 0$, $F_1 = 01$, and $F_{n+2} = F_{n+1}F_n$, $n \in \mathbb{N}_0$. The *Fibonacci word* F is then defined (see e.g. [Py02]) as the sequence $F := 0100101001\dots$, whose prefixes are the F_n, and is a Sturmian word (see e.g. [Py02]) of parameter $(3-\sqrt{5})/2$. There exists $\beta \in (1,2)$ such that $\pi_\beta(1) = 1F$, i.e., the concatenation of 1 and F (cf. e.g. [CK04, p. 404]). Clearly, β is not a simple beta-number. Since $(\overline{\mathcal{O}^\sigma(1F)}, \sigma)$ is minimal (cf. [CK04, pp. 397–398]), β is emergent.

Lemma 6.5. *If* $\beta > 1$ *is emergent then the restriction* $U_\beta|_{\overline{\mathcal{O}'_\beta(1)}}$ *is continuous, and the dynamical system* $\big(\overline{\mathcal{O}'_\beta(1)}, U_\beta\big)$ *is minimal.*

Proof. Since β is emergent, then $0 \notin \overline{\mathcal{O}'_\beta(1)}$ by Proposition 6.3. Write $\delta := d\big(0, \overline{\mathcal{O}'_\beta(1)}\big)$. By (3.2), $\overline{\mathcal{O}'_\beta(1)} \cap (y, y+\delta/\beta) = \emptyset$ for each $y \in D_\beta$. So for each pair of $x, y \in \overline{\mathcal{O}'_\beta(1)}$ with $|x-y| < \delta/\beta$, we have $(x,y) \cap D_\beta = \emptyset$ and hence $U_\beta(x) - U_\beta(y) = \beta(x-y)$. The first part follows.

Let $Y \subseteq \overline{\mathcal{O}'_\beta(1)}$ be an arbitrary non-empty closed subset of $\overline{\mathcal{O}'_\beta(1)}$ with $U_\beta(Y) = Y$. If $1 \in Y$, then $Y = \overline{\mathcal{O}'_\beta(1)}$. So to show that $\big(\overline{\mathcal{O}'_\beta(1)}, U_\beta\big)$ is minimal, it suffices to show that 1 must belong to Y. If on the contrary $1 \notin Y$, then by Lemma 3.20 there exists $\gamma \in (1,\beta)$ such that $Y \subseteq H_\beta^\gamma$. But β is emergent, so $Y \subseteq \overline{\mathcal{O}'_\beta(1)} \cap H_\beta^\gamma \subseteq Z_\beta$ by Proposition 6.3. By Lemma 4.2 (i), for each $y \in Y$, there exists $n \in \mathbb{N}$ with $U_\beta^n(y) = 1$, which contradicts the assumption that $1 \notin Y$. The lemma follows. □

By the following result, emergent numbers constitute a small subset of $(1,+\infty)$.

Corollary 6.6. *The emergent set* $\mathcal{E}$ *has zero Lebesgue measure, and is a meagre subset of* $(1,+\infty)$*.*

Proof. Suppose $\beta \in \mathcal{E}$. By Proposition 6.3, $\big(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\big)$ is minimal, so the fixed point $(0)^\infty$ does not belong to $\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}$, and therefore the beta-shift $(\mathcal{S}_\beta, \sigma)$ is a specified system (see [Bl89, Proposition 4.5], [Sc97, Proposition 3.5]). So if Spec denotes the set of those $\beta > 1$ such that $(\mathcal{S}_\beta, \sigma)$ is specified, then $\mathcal{E} \subseteq$ Spec. But Spec is a meagre subset of $(1, +\infty)$ by [Sc97, Theorem B], and has zero Lebesgue measure by [Sc97, Theorem E], therefore $\mathcal{E}$ also has these properties. $\square$

7. TYPICAL PERIODIC OPTIMIZATION

The main goal of this section is to prove the theorems stated in Section 1. Our proofs rely on the properties of two important sets, $\mathrm{Crit}^\alpha(\beta)$ and $\mathcal{R}^\alpha(\beta)$. Some characterisations of the set $\mathrm{Crit}^\alpha(\beta)$, consisting of functions where the critical orbit is maximizing, are given in Subsection 7.1. In Subsection 7.2, we first define the local (periodic) locking set $\mathcal{R}^\alpha(\beta) \subseteq C^{0,\alpha}(I)$ which consists of $\phi \in C^{0,\alpha}(I)$ satisfying $\phi|_{H_\beta^\gamma} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^\gamma}\big)$ for all simple beta-numbers $\gamma \in (1, \beta)$. We then prove that $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$ (Proposition 7.8). Next we prove Theorems 1.2 and 1.4 in Subsections 7.3 and 7.4. Theorem 1.1 follows directly from Theorems 1.2 and 1.4 (see Subsection 7.5).

7.1. **The set** $\mathrm{Crit}^\alpha(\beta)$.

Definition 7.1. For $\beta > 1$ and $\alpha \in (0, 1]$, define

$$\mathrm{Crit}^\alpha(\beta) \coloneqq \big\{\phi \in C^{0,\alpha}(I) : \mathcal{O}'_\beta(1) \text{ is a maximizing orbit for } (U_\beta, \phi)\big\}. \tag{7.1}$$

This naturally prompts an investigation into those invariant measures that are generated by the orbit of the point 1. It is convenient to first define the following notions in X_β:

Definition 7.2. For $\beta > 1$, define the *empirical measure* μ_n on X_β by

$$\mu_n \coloneqq \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i(\pi_\beta^*(1))}.$$

A measure $\mu \in \mathcal{M}(X_\beta, \sigma)$ is said to be *quasi-generated* by $\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)$ if it is an accumulation point of the sequence $\{\mu_n\}_{n \in \mathbb{N}}$. Let $\mathrm{QG}(\beta)$ denote the set of measures that are quasi-generated by $\mathcal{O}^\sigma(\pi_\beta^*(1))$ and let $\mathrm{CQG}(\beta)$ denote the convex hull of $\mathrm{QG}(\beta)$.

Clearly $\mathrm{QG}(\beta) \subseteq \mathcal{M}\big(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\big)$, and hence $\mathrm{CQG}(\beta) \subseteq \mathcal{M}\big(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\big)$. The set $\mathrm{QG}(\beta)$ is known to be weak* closed (see [DGS76, Proposition 3.8] and [Gla03, p. 98]), hence $\mathrm{CQG}(\beta)$ is weak* closed.

The above notions lead to an alternative expression for $\mathrm{Crit}^\alpha(\beta)$:

Lemma 7.3. *Suppose $\beta > 1$ and $\alpha \in (0, 1]$. Then*

$$\mathrm{Crit}^\alpha(\beta) = \big\{\phi \in C^{0,\alpha}(I) : H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)\big\}. \tag{7.2}$$

Moreover, $\mathrm{Crit}^\alpha(\beta)$ is a closed subset of $C^{0,\alpha}(I)$.

Proof. We first verify (7.2). By (7.1), Proposition 3.9 (v), and Proposition 4.3 (iii),

$$\phi \in \mathrm{Crit}^\alpha(\beta) \text{ if and only if } \mathcal{O}^\sigma\big(\pi_\beta^*(1)\big) \text{ is } (\sigma|_{X_\beta}, \phi \circ h_\beta)\text{-maximizing}. \tag{7.3}$$

Assume that $\phi \in \mathrm{Crit}^\alpha(\beta)$. Then (7.3) implies that

$$\lim_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)\big(\pi_\beta^*(1)\big) = Q(\sigma|_{X_\beta}, \phi \circ h_\beta),$$

and therefore $\mathrm{QG}(\beta) \subseteq \mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)$. Hence $\mathrm{CQG}(\beta) \subseteq \mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)$. By Proposition 4.3 (i) and (iii), it follows that $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$.

Conversely, assume that $\phi \notin \mathrm{Crit}^\alpha(\beta)$. By [Je19, Theorem 2.2],

$$\limsup_{n\to+\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)\bigl(\pi_\beta^*(1)\bigr) \leqslant Q(\sigma|_{X_\beta}, \phi \circ h_\beta),$$

but the fact that $\phi \notin \mathrm{Crit}^\alpha(\beta)$ means there exists a subsequence $n_k \nearrow +\infty$ such that

$$\lim_{k\to+\infty} \frac{1}{n_k} S_{n_k}^\sigma(\phi \circ h_\beta)\bigl(\pi_\beta^*(1)\bigr) < Q(\sigma|_{X_\beta}, \phi \circ h_\beta).$$

Hence any accumulation point of the sequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$, which belongs to $\mathrm{QG}(\beta)$ by Definition 7.2, cannot belong to $\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi\circ h_\beta)$. By Proposition 4.3 (i) and (iii), it follows that $H_\beta(\mathrm{CQG}(\beta))$ is not contained in $\mathcal{M}_{\max}(U_\beta, \phi)$.

We then prove that $\mathrm{Crit}^\alpha(\beta)$ is a closed subset of $C^{0,\alpha}(I)$. If $\phi_n \in \mathrm{Crit}^\alpha(\beta)$, then

$$\int \phi_n \,\mathrm{d}\mu = Q(U_\beta, \phi_n) \tag{7.4}$$

for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$, by (7.2). If $\phi_n \to \phi$ in $(C^{0,\alpha}(I), \|\cdot\|_\alpha)$, then $\int \phi_n \,\mathrm{d}\mu \to \int \phi \,\mathrm{d}\mu$ for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$. But $Q(U_\beta, \cdot)$ is (1-Lipschitz) continuous, so $Q(U_\beta, \phi_n) \to Q(U_\beta, \phi)$, and therefore (7.4) implies that $\int \phi \, d\mu = Q(U_\beta, \phi)$ for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$. So $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$. Hence $\phi \in \mathrm{Crit}^\alpha(\beta)$ by (7.2), and therefore $\mathrm{Crit}^\alpha(\beta)$ is closed. □

If $\beta > 1$ is emergent, the following lemma gives another characterisation of $\mathrm{Crit}^\alpha(\beta)$.

Lemma 7.4. *Assume that $\beta > 1$ is emergent, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$. Then the following are equivalent:*

- (i) $\phi \in \mathrm{Crit}^\alpha(\beta)$.
- (ii) $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$.
- (iii) $1 \in \operatorname{supp}\mu$ *for some* $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$.

Proof. By Lemma 7.3, (i) is equivalent to (ii), so it suffices to prove that (i) is equivalent to (iii). For this, first assume that $\phi \in \mathrm{Crit}^\alpha(\beta)$, so that $\mathcal{O}'_\beta(1)$ is (U_β, ϕ)-maximizing. Suppose that μ is any accumulation point of $\bigl\{\frac{1}{n}\sum_{i=0}^{n-1} \delta_{U_\beta^i(1)}\bigr\}_{n\in\mathbb{N}}$. Note that $\overline{\mathcal{O}'_\beta(1)}$ is compact, $U_\beta\bigl(\overline{\mathcal{O}'_\beta(1)}\bigr) = \overline{\mathcal{O}'_\beta(1)}$, and $U_\beta|_{\overline{\mathcal{O}'_\beta(1)}}$ is continuous (see Lemma 6.5). We obtain that $\operatorname{supp}\mu \subseteq \overline{\mathcal{O}'_\beta(1)}$ and $\mu \in \mathcal{M}\bigl(\overline{\mathcal{O}'_\beta(1)}, U_\beta|_{\overline{\mathcal{O}'_\beta(1)}}\bigr)$ (see [Wa82, Theorem 6.9]). Hence μ is also U_β-invariant as a measure on I. In addition, $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ since $\mathcal{O}'_\beta(1)$ is (U_β, ϕ)-maximizing. But β is emergent, so $\bigl(\overline{\mathcal{O}'_\beta(1)}, U_\beta\bigr)$ is minimal by Lemma 6.5. Note that μ can be seen as a measure on $\overline{\mathcal{O}'_\beta(1)}$ and it follows from [Ak93, p. 156] that $U_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$. Hence $\operatorname{supp}\mu$ must equal $\overline{\mathcal{O}'_\beta(1)}$, and in particular $1 \in \operatorname{supp}\mu$.

Conversely, if we assume that $1 \in \operatorname{supp}\mu$ for some $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$, then (5.45), Theorem 5.9, 5.10 (ii), and Lemma 5.2 (iii) together imply that $\mathcal{O}'_\beta(1)$ is a maximizing orbit for (U_β, ϕ), and hence that $\phi \in \mathrm{Crit}^\alpha(\beta)$. □

Lemma 7.5. *Suppose $\beta > 1$, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$. If $1 \notin \operatorname{supp}\mu$ for some $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$, then there exists $\beta' \in (1, \beta)$ such that $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ for all $\gamma \in (\beta', \beta)$.*

Proof. Suppose $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ and $1 \notin \operatorname{supp}\mu =: \mathcal{K}$. It follows that $\mu \in \mathcal{M}(I, T_\beta)$ by Proposition 3.13 (iii), and $T_\beta(\mathcal{K}) = \mathcal{K}$ by Lemma 3.14. Therefore, by Lemma 3.20, there exists $\beta' \in (1, \beta)$ such that $\mathcal{K} \subseteq H_\beta^\gamma$ for all $\gamma \in (\beta', \beta)$. In particular, μ can be considered as an element of $\mathcal{M}\big(H_\beta^\gamma, T_\beta\big)$ for all $\gamma \in (\beta', \beta)$, and thus $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ by (3.14), as required. □

If β is emergent, we have the following corollary:

Corollary 7.6. *Suppose $\beta > 1$ is emergent, $\alpha \in (0,1]$, and $\phi \in C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$. Then there exists $\beta' \in (1, \beta)$ such that $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ for all $\gamma \in (\beta', \beta)$.*

Proof. Proposition 4.3 (iii) implies $\mathcal{M}_{\max}(U_\beta, \phi) \neq \emptyset$. Since $\phi \notin \mathrm{Crit}^\alpha(\beta)$, Lemma 7.4 implies that if $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ then $1 \notin \operatorname{supp} \mu$. The corollary now follows from Lemma 7.5. □

7.2. **The set $\mathcal{R}^\alpha(\beta)$.** In this subsection, we use the approximation method developed in Subsection 3.2 to prove a key result (Proposition 7.8).

Definition 7.7. For each $\beta > 1$, denote the set of simple beta-numbers contained in $(1, \beta)$ by $\mathrm{Sim}(\beta)$.

For each $\beta > 1$ and each $\alpha \in (0,1]$, we define $\mathcal{R}^\alpha(\beta) \subseteq C^{0,\alpha}(I)$ by

$$\mathcal{R}^\alpha(\beta) \coloneqq \big\{\phi \in C^{0,\alpha}(I) : \phi|_{H_\beta^\gamma} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^\gamma}\big) \text{ for all } \gamma \in \mathrm{Sim}(\beta)\big\}.$$

Proposition 7.8. *For all $\beta > 1$, $\alpha \in (0,1]$, the set $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$.*

To prove Proposition 7.8, we mainly use the approximation method. For each $\beta \in (1, +\infty)$, by [Pa60, Theorem 5], $\mathrm{Sim}(\beta)$ is dense in $(1, \beta)$. By Definition 3.10, $\gamma \in \mathrm{Sim}(\beta)$ if and only if $\pi_\beta(1)$ has finitely many non-zero terms and $\gamma \in (1, \beta)$. Note that the function i_1, as defined in Definition 3.7, is by Proposition 3.9 (xiii) a strictly increasing map from $(1, \beta)$ to $\mathcal{B}^{\mathbb{N}}$. Hence we obtain that $\mathrm{Sim}(\beta)$ is countable. Let us enumerate $\mathrm{Sim}(\beta) = \{\gamma_1, \dots, \gamma_n, \dots\}$. Then for each $\phi \in C^{0,\alpha}(I)$, we use Theorem 7.9 below to get a sequence of functions $\{\phi_n\}_{n\in\mathbb{N}}$ in $C^{0,\alpha}(I)$ sufficiently close to ϕ such that $\phi_n|_{H_\beta^{\gamma_n}} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^{\gamma_n}}\big)$ for each $n \in \mathbb{N}$. Next, we check that $\{\phi_n\}_{n\in\mathbb{N}}$ converges to a function $\phi_\infty \in C^{0,\alpha}(I)$ near ϕ.

Another crucial ingredient in the proof of Proposition 7.8 is the following typical periodic optimization theorem due to Contreras [Co16, Theorem A] (the statement below is closer to the reformulation of Bochi [Boc19]):

Theorem 7.9 ([Co16])**.** *If X is a compact metric space and $T\colon X \to X$ is a Lipschitz continuous open distance-expanding map, and $\alpha \in (0,1]$, then* $\mathrm{Lock}^\alpha(T)$ *is an open and dense subset of $C^{0,\alpha}(X)$.*

Remark 7.10. The definition of *expanding* used in [Co16] is precisely what we here call *open distance-expanding and Lipschitz continuous.* Note that Contreras' result has been extended in [HLMXZ19] to uniformly hyperbolic systems.

In the proof of Proposition 7.8 below, we will need to extend a Hölder function, defined on a subset of I, to the whole of I, without increasing its norm. The following lemma guarantees that this can be done.

Lemma 7.11. *For $\alpha \in (0,1]$, $\emptyset \neq \mathcal{K} \subseteq I$, and $\phi \in C^{0,\alpha}(\mathcal{K})$, there exists $\psi \in C^{0,\alpha}(I)$ such that $\psi|_{\mathcal{K}} = \phi$ and $\|\psi\|_{\alpha,I} = \|\phi\|_{\alpha,\mathcal{K}}$.*

Proof. It follows immediately from the McShane extension theorem ([We18, Theorem 1.33]). □

Proof of Proposition 7.8. Let $\epsilon > 0$ and $\phi \in C^{0,\alpha}(I)$ be arbitrary. Since $\mathrm{Sim}(\beta)$ is countable, we write

$$\mathrm{Sim}(\beta) = \{\gamma_1, \dots, \gamma_n, \dots\}.$$

We will recursively construct a sequence of functions $\{\phi_n\}_{n\in\mathbb{N}}$ in $C^{0,\alpha}(I)$, and two sequences of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{e_n\}_{n\in\mathbb{N}}$ below:

Base step. Define $\phi_0 \coloneqq \phi$ and $\delta_0 \coloneqq 1$.

Recursive step. For $n \in \mathbb{N}$, assume that $\phi_0, \dots, \phi_{n-1}, \delta_0, \dots, \delta_{n-1}$ are defined. Define

$$e_n \coloneqq \min\{2^{-n}\epsilon,\, 2^{-n}\delta_0,\, \dots,\, 2^{-1}\delta_{n-1}\}. \tag{7.5}$$

Since γ_n is a simple beta-number, Proposition 3.22 gives that $T_\beta|_{H_\beta^{\gamma_n}}$ is Lipschitz continuous, open, and distance-expanding, and then Contreras' Theorem 7.9 guarantees that there exists $\psi_n \in C^{0,\alpha}\bigl(H_\beta^{\gamma_n}\bigr)$ satisfying

$$\psi_n \in \operatorname{Lock}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_n}}\bigr) \text{ and } \bigl\|\phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n\bigr\|_{\alpha,H_\beta^{\gamma_n}} < e_n.$$

Applying Lemma 7.11, for the subset $H_\beta^{\gamma_n} \subseteq I$ and function $\phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n$, we obtain an α-Hölder extension function Φ_n defined on all of I, and with the same α-Hölder norm, in other words, there exists $\Phi_n \in C^{0,\alpha}(I)$ satisfying $\Phi_n|_{H_\beta^{\gamma_n}} = \phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n$ and $\|\Phi_n\|_{\alpha,I} < e_n$. Defining $\phi_n \coloneqq \phi_{n-1} - \Phi_n$ then gives

$$\phi_n|_{H_\beta^{\gamma_n}} = \psi_n \text{ and } \|\phi_{n-1} - \phi_n\|_{\alpha,I} < e_n. \tag{7.6}$$

Finally, δ_n is defined to be any value such that

$$\bigl\{\Phi \in C^{0,\alpha}\bigl(H_\beta^{\gamma_n}\bigr) : \bigl\|\Phi - \phi_n|_{H_\beta^{\gamma_n}}\bigr\|_{\alpha,H_\beta^{\gamma_n}} < \delta_n\bigr\} \subseteq \operatorname{Lock}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_n}}\bigr), \tag{7.7}$$

where again Theorem 7.9 and Proposition 3.22 guarantee that such a δ_n exists. Since each of e_n, ϕ_n, and δ_n have been defined, the recursive step is complete.

By (7.6) and (7.5), $\{\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to some $\phi_\infty \in C^{0,\alpha}(I)$, and

$$\|\phi - \phi_\infty\|_{\alpha,I} \leqslant \sum_{n=1}^{+\infty} \|\phi_{n-1} - \phi_n\|_{\alpha,I} < \sum_{n=1}^{+\infty} e_n \leqslant \epsilon. \tag{7.8}$$

For each $n \in \mathbb{N}$, from (7.5) it follows that $\sum_{i=n+1}^{+\infty} e_i \leqslant \delta_n$ and then

$$\bigl\|\phi_n|_{H_\beta^{\gamma_n}} - \phi_\infty|_{H_\beta^{\gamma_n}}\bigr\|_{\alpha,H_\beta^{\gamma_n}} \leqslant \|\phi_n - \phi_\infty\|_{\alpha,I} \leqslant \sum_{i=n}^{+\infty} \|\phi_i - \phi_{i+1}\|_{\alpha,I} < \sum_{i=n+1}^{+\infty} e_i \leqslant \delta_n. \tag{7.9}$$

But (7.7) and (7.9) mean that $\phi_\infty|_{H_\beta^{\gamma_n}} \in \operatorname{Lock}^\alpha(T_\beta|_{H_\beta^{\gamma_n}})$ for all $n \in \mathbb{N}$, in other words, $\phi_\infty \in \mathcal{R}^\alpha(\beta)$. But $\epsilon > 0$ was arbitrary, so the result follows. □

Corollary 7.12. *For all $\beta > 1$ and $\alpha \in (0,1]$, the set $\mathcal{R}^\alpha(\beta) \smallsetminus \operatorname{Crit}^\alpha(\beta)$ is a dense subset of $C^{0,\alpha}(I) \smallsetminus \operatorname{Crit}^\alpha(\beta)$.*

Proof. It follows immediately from the fact that $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$ by Proposition 7.8, and $\operatorname{Crit}^\alpha(\beta)$ is closed in $C^{0,\alpha}(I)$ by Lemma 7.3. □

7.3. **Proof of Theorem 1.4.**

Corollary 7.13. *If $\beta > 1$ is emergent and $\alpha \in (0,1]$, then*

$$\mathcal{R}^\alpha(\beta) \smallsetminus \operatorname{Crit}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(\beta) \smallsetminus \operatorname{Crit}^\alpha(\beta). \tag{7.10}$$

Proof. Suppose $\phi \in \mathcal{R}^\alpha(\beta) \smallsetminus \operatorname{Crit}^\alpha(\beta)$. By Corollary 7.6, there is a simple beta-number $\gamma_0 \in (1,\beta)$ such that

$$Q_{\beta,\gamma_0}(\phi) = Q(U_\beta, \phi). \tag{7.11}$$

The fact that $\phi \in \mathcal{R}^\alpha(\beta)$ implies that

$$\phi|_{H_\beta^{\gamma_0}} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big) \subseteq \mathscr{P}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big).$$

So there exists a periodic measure $\mu \in \mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$ (see Proposition 3.13 (iii)) such that

$$\int \phi \, \mathrm{d}\mu = Q_{\beta,\gamma_0}(\phi). \tag{7.12}$$

So from (7.11) and (7.12) we see that the periodic measure μ satisfies $\int \phi \, \mathrm{d}\mu = Q(U_\beta, \phi)$, and therefore $\phi \in \mathscr{P}^\alpha(\beta)$. But $\phi \notin \mathrm{Crit}^\alpha(\beta)$, so (7.10) follows. $\square$

We require the following result, whose proof will be deferred to Appendix A.

Theorem A.1. *If $\beta > 1$ and $\alpha \in (0, 1]$, then the set* $\mathrm{Lock}^\alpha(\beta)$ *is an open and dense subset of $\mathscr{P}^\alpha(\beta)$.*

Recall that Theorem 1.4 asserts:

Theorem 1.4. *If $\beta > 1$ is emergent and $\alpha \in (0, 1]$, then $C^{0,\alpha}(I)$ is equal to the union of* $\mathrm{Crit}^\alpha(\beta)$ *and the closure of the open set* $\mathrm{Lock}^\alpha(\beta)$.

Proof of Theorem 1.4. Fix an arbitrary $\beta > 1$ that is emergent and $\alpha \in (0, 1]$. We wish to show that $\mathrm{Lock}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$. By Theorem A.1 and the fact that $\mathrm{Crit}^\alpha(\beta)$ is closed (see Lemma 7.3), it suffices to prove that $\mathscr{P}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$. The set $\mathcal{R}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ is a subset of $\mathscr{P}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ by Corollary 7.13, and is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$, by Corollary 7.12. Therefore $\mathscr{P}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ is itself dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$, as required. $\square$

7.4. **Proof of Theorem 1.2.**

Lemma 7.14. *Suppose that $\beta > 1$ is non-emergent, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$. Then there exists $\beta' \in (1, \beta)$ such that $Q_{\beta,\beta'}(\phi) = Q(U_\beta, \phi)$.*

Proof. By Proposition 4.3 (iii) there exists $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$. Let us denote $\mathcal{K} := \operatorname{supp} \mu$.

If $0 \in \mathcal{K}$, then $\widetilde{\phi}^+(0) = 0 = Q\big(U_\beta, \widetilde{\phi}^+\big)$ or $\widetilde{\phi}^-(0) = 0 = Q\big(U_\beta, \widetilde{\phi}^-\big)$ by Theorems 5.10 (i) and 5.9 (ii), so $\delta_0 \in \mathcal{M}_{\max}(U_\beta, \phi)$ (by (5.46) and Theorem 5.9 (i)), and therefore $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ for every $\gamma \in (1, \beta)$.

If $0 \notin \mathcal{K}$ then $U_\beta(\mathcal{K}) = \mathcal{K}$ by Lemma 3.14. Let us assume, for a contradiction, that the result is false, i.e., that

$$Q_{\beta,\gamma}(\phi) < Q(U_\beta, \phi) \text{ for all } \gamma \in (1, \beta). \tag{7.13}$$

Lemma 7.5 then implies that $1 \in \mathcal{K}$. Since β is non-emergent, Proposition 6.3 implies that there exists $\gamma \in (1, \beta)$ such that $\overline{\mathcal{O}'_\beta(1)} \cap H_\beta^\gamma$ is not a subset of Z_β, where we recall from (4.1) that $Z_\beta = \{x \in I : \pi_\beta(x) \neq \pi_\beta^*(x)\}$. But $1 \in \mathcal{K}$, so $\overline{\mathcal{O}'_\beta(1)} \subseteq \mathcal{K}$, and therefore $\mathcal{K} \cap H_\beta^\gamma$ is not a subset of Z_β, in other words, there exists $x \in \big(\mathcal{K} \cap H_\beta^\gamma\big) \smallsetminus Z_\beta$.

By Lemma 4.2 (iii), the orbit $\mathcal{O}_\beta(x)$ is equal to $\mathcal{O}'_\beta(x)$, and it is contained in $\mathcal{K} \cap H_\beta^\gamma$ since $U_\beta(\mathcal{K}) = \mathcal{K}$ and $T_\beta(H_\beta^\gamma) \subseteq H_\beta^\gamma$. By Theorem 5.10 (i), $\widetilde{\phi}^+|_{\mathcal{O}_\beta(x)} \equiv 0$ or $\widetilde{\phi}^-|_{\mathcal{O}_\beta(x)} \equiv 0$. In other words, $\mathcal{O}_\beta(x)$ is contained in either $(\widetilde{\phi}^-)^{-1}(0)$ or $(\widetilde{\phi}^+)^{-1}(0)$. Since $\widetilde{\phi}^- \leqslant 0$ and $\widetilde{\phi}^+ \leqslant 0$ (by Theorem 5.9 (ii)), Lemma 5.2 (iii) implies that $\mathcal{O}_\beta(x) = \mathcal{O}'_\beta(x)$ is a (T_β, ϕ)-maximizing orbit and a (U_β, ϕ)-maximizing orbit (by Proposition 4.4 (ii)), so in particular,

$$\lim_{n \to +\infty} \frac{1}{n} S_n \phi(x) = Q(U_\beta, \phi). \tag{7.14}$$

Now $\mathcal{O}_\beta(x) \subseteq H_\beta^\gamma$ and $T_\beta|_{H_\beta^\gamma}$ is continuous (see Proposition 3.22 (i)), so by [Je19, Theorem 2.2], the corresponding time average is bounded above by the maximum ergodic average, in other words,

$$Q_{\beta,\gamma}(\phi) \geqslant \lim_{n\to+\infty} \frac{1}{n} S_n\phi(x). \tag{7.15}$$

Now as an immediate consequence of (3.14), we have $Q(U_\beta,\phi) \geqslant Q_{\beta,\gamma}(\phi)$. So combining this inequality with (7.14) and (7.15) gives $Q(U_\beta,\phi) = Q_{\beta,\gamma}(\phi)$, which gives the required contradiction to (7.13). The lemma follows. ☐

Corollary 7.15. *Suppose $\beta > 1$ is non-emergent, $\alpha \in (0,1]$, and $\phi \in C^{0,\alpha}(I)$. Then there exists $\beta' \in (1,\beta)$ such that*

$$Q_{\beta,\gamma}(\phi) = Q\bigl(U_\beta,\phi\bigr) \quad \textit{for all } \gamma \in [\beta',\beta). \tag{7.16}$$

Proof. Let β' be as in Lemma 7.14. If $\gamma \in [\beta',\beta)$ then

$$Q_{\beta,\beta'}(\phi) \leqslant Q_{\beta,\gamma}(\phi) \leqslant Q(U_\beta,\phi), \tag{7.17}$$

an immediate consequence of (3.14), since $H_\beta^{\beta'} \subseteq H_\beta^\gamma \subseteq I$. But $Q(U_\beta,\phi) = Q_{\beta,\beta'}(\phi)$, by Lemma 7.14, so (7.17) implies the required equality (7.16). ☐

The proof of the following result is similar to that of Corollary 7.13:

Corollary 7.16. *If $\beta > 1$ is non-emergent, then $\mathcal{R}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(\beta)$.*

Proof. Suppose $\phi \in \mathcal{R}^\alpha(\beta)$. By Corollary 7.15, there exists $\beta' \in (1,\beta)$ such that $Q_{\beta,\gamma}(\phi) = Q\bigl(U_\beta,\phi\bigr)$ for all $\gamma \in [\beta',\beta)$, so in particular there is a simple beta-number $\gamma_0 \in (1,\beta)$ such that

$$Q_{\beta,\gamma_0}(\phi) = Q(U_\beta,\phi). \tag{7.18}$$

The fact that $\phi \in \mathcal{R}^\alpha(\beta)$ means that

$$\phi|_{H_\beta^{\gamma_0}} \in \operatorname{Lock}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_0}}\bigr) \subseteq \mathscr{P}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_0}}\bigr).$$

So there exists a periodic measure $\mu \in \mathcal{M}(I,T_\beta) \subseteq \mathcal{M}(I,U_\beta)$ (see Proposition 3.13 (iii)) such that

$$\int \phi \,\mathrm{d}\mu = Q_{\beta,\gamma_0}(\phi). \tag{7.19}$$

Thus from (7.18), (7.19) we see that the periodic measure μ satisfies $\int \phi \,\mathrm{d}\mu = Q(U_\beta,\phi)$. Therefore $\phi \in \mathscr{P}^\alpha(\beta)$, as required. ☐

Recall that Theorem 1.2 asserts:

Theorem 1.2. *If $\beta > 1$ is non-emergent and $\alpha \in (0,1]$, then $\operatorname{Lock}^\alpha(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Proof of Theorem 1.2. Now $\operatorname{Lock}^\alpha(\beta)$ is by definition an open subset of $C^{0,\alpha}(I)$, and Theorem A.1 asserts that $\operatorname{Lock}^\alpha(\beta)$ is dense in $\mathscr{P}^\alpha(\beta)$, so it suffices to prove that $\mathscr{P}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$. Since $\beta > 1$ is non-emergent, Corollary 7.16 gives $\mathcal{R}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(\beta)$, and $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$ by Proposition 7.8. So it follows that $\mathscr{P}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$, as required. ☐

Recall that Corollary 1.3 asserts:

Corollary 1.3. *Fix $\alpha \in (0,1]$. For a set of values $\beta > 1$ which is both residual and of full Lebesgue measure, the periodic locking set $\operatorname{Lock}^\alpha(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Proof of Corollary 1.3. By Corollary 6.6, the set $(1,+\infty)\smallsetminus\mathcal{E}$ of non-emergent numbers has zero Lebesgue measure, and is a residual subset of $(1,+\infty)$. By Theorem 1.2, if $\beta\in(1,+\infty)\smallsetminus\mathcal{E}$ then $\mathrm{Lock}^\alpha(\beta)$ is open and dense in $C^{0,\alpha}(I)$, so the result follows. □

7.5. **Proof of Theorem 1.1.** Recall that Theorem 1.1 asserts:

Theorem 1.1. *For a beta-number $\beta>1$ and any $\alpha\in(0,1]$, the set $\mathrm{Lock}^\alpha(\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Proof of Theorem 1.1. Assume that β is a non-simple beta-number. Then $\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)$ is preperiodic but not periodic. So $\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}=\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)$ and $\big(\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big),\sigma\big)$ is not minimal. Hence β is non-emergent by Proposition 6.3. So the result follows from Theorem 1.2.

Assume that β is a simple beta-number. Then $\mathcal{O}'_\beta(1)$ and $\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)$ are periodic, so $\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}=\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big)$ and $\big(\mathcal{O}^\sigma\big(\pi_\beta^*(1)\big),\sigma\big)$ is minimal. Hence β is emergent by Proposition 6.3. Theorem 1.4 gives that $C^{0,\alpha}(I)$ is equal to the union of $\mathrm{Crit}^\alpha(\beta)$ and the closure of $\mathrm{Lock}^\alpha(\beta)$, so it suffices to show that $\mathrm{Crit}^\alpha(\beta)$ is a subset of the closure of $\mathrm{Lock}^\alpha(\beta)$. If $\phi\in\mathrm{Crit}^\alpha(\beta)$ then from (7.2) we see that the periodic measure supported by $\mathcal{O}'_\beta(1)$ is (U_β,ϕ)-maximizing, so $\phi\in\mathscr{P}^\alpha(\beta)$, and therefore ϕ belongs to the closure of $\mathrm{Lock}^\alpha(\beta)$ by Theorem A.1. Theorem 1.1 follows. □

Appendix A. Periodic locking and periodic optimization

The purpose of this appendix is to consider the set $\mathrm{Lock}^\alpha(\beta)\subseteq\mathscr{P}^\alpha(\beta)$ (see Section 1) and prove the following:

Theorem A.1. *If $\beta>1$ and $\alpha\in(0,1]$, then the set $\mathrm{Lock}^\alpha(\beta)$ is an open and dense subset of $\mathscr{P}^\alpha(\beta)$.*

A statement analogous to Theorem A.1 appeared as Remark 4.5 in [YH99] for maps with hyperbolicity, and as Proposition 1 in the unpublished note [BZ15] for continuous maps. In view of the slightly different setting here, and the folklore status of this result, a full proof, using ideas from [BZ15], is included for the convenience of the reader.

Lemma A.2. *Suppose $\beta>1$, $\alpha\in(0,1]$, and let $\mu\in\mathcal{M}(I,U_\beta)$ be supported on a periodic orbit $\mathcal{O}'$ of U_β. Then there exists $C_\mu>0$ such that for all $\nu\in\mathcal{M}(I,U_\beta)$ and $\phi\in C^{0,\alpha}(I)$,*

$$\int_I\phi\,\mathrm{d}\nu\leqslant\int_I\phi\,\mathrm{d}\mu+C_\mu|\phi|_{\alpha,I}\int_I d(\cdot,\mathcal{O}')^\alpha\,\mathrm{d}\nu. \tag{A.1}$$

Proof. Let us write $n:=\operatorname{card}\mathcal{O}'$.

Case 1. If $n=1$ then $\mathcal{O}'=\{y\}$ for some $y\in I$, and (A.1) holds with $C_\mu=1$ because $\int_I\phi\,\mathrm{d}\nu\leqslant\int_I(\phi(y)+|\phi|_{\alpha,I}d(\cdot,y)^\alpha)\,\mathrm{d}\nu=\int_I\phi\,\mathrm{d}\mu+|\phi|_{\alpha,I}\int_I d(\cdot,\mathcal{O}')^\alpha\,\mathrm{d}\nu$,

Case 2. If $n>1$ then by ergodic decomposition, it suffices to prove (A.1) for every ergodic $\nu\in\mathcal{M}(I,U_\beta)$. Fixing an arbitrary ergodic $\nu\in\mathcal{M}(I,U_\beta)$, the ergodic theorem implies that there exists $a\in I$ with

$$\int_I\phi\,\mathrm{d}\nu=\lim_{k\to+\infty}\frac{1}{k}S_k^{U_\beta}\phi(a)\quad\text{and} \tag{A.2}$$

$$\int_I d(\cdot,\mathcal{O}')^\alpha\,\mathrm{d}\nu=\lim_{k\to+\infty}\frac{1}{k}\sum_{i=0}^{k-1}d\big(U_\beta^i(a),\mathcal{O}'\big)^\alpha. \tag{A.3}$$

Claim. There exists $C_\mu > 0$ and a sequence $\underline{y} = \{y_i\}_{i=-1}^{+\infty}$ with entries from $\mathcal{O}'$ such that

$$\lim_{k\to+\infty} \frac{1}{k} \operatorname{card}\{i \in [0, k-1] \cap \mathbb{N}_0 : y_i = y\} = \frac{1}{n} \quad \text{for each } y \in \mathcal{O}' \text{ and} \tag{A.4}$$

$$\bigl|U_\beta^i(a) - y_i\bigr| \leqslant C_\mu^{1/\alpha} d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) \quad \text{for each } i \in \mathbb{N}. \tag{A.5}$$

Note that a consequence of this Claim is, by (A.2), (A.5), (A.4), and (A.3), that

$$\begin{aligned}\int_I \phi \,\mathrm{d}\nu &= \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} \phi\bigl(U_\beta^i(a)\bigr)\\ &\leqslant \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} \Bigl(\phi(y_i) + |\phi|_{\alpha,I} \bigl|U_\beta^i(a) - y_i\bigr|^\alpha\Bigr)\\ &\leqslant \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} \Bigl(\phi(y_i) + C_\mu |\phi|_{\alpha,I} d\bigl(U_\beta^i(a), \mathcal{O}'\bigr)^\alpha\Bigr)\\ &= \frac{1}{n} \sum_{y\in\mathcal{O}'} \phi(y) + C_\mu |\phi|_{\alpha,I} \lim_{k\to+\infty} \frac{1}{k} \sum_{i=0}^{k-1} d\bigl(U_\beta^i(a), \mathcal{O}'\bigr)^\alpha\\ &= \int_I \phi \,\mathrm{d}\mu + C_\mu |\phi|_{\alpha,I} \int_I d\bigl(\cdot, \mathcal{O}'\bigr)^\alpha \,\mathrm{d}\nu.\end{aligned}$$

So the required inequality (A.1) will hold, and the lemma will follow.

Proof of Claim. Our discussion will be divided into 2 cases. In each case it will be convenient to write $\Delta(\mathcal{O}') := \min\{d(x,y) : x, y \in \mathcal{O}', \, x \neq y\}$.

Case 1. Assume that $\mathcal{O}'$ is not the orbit of 1 under U_β. By Proposition 3.13 (i), we have $D_\beta \cap \mathcal{O}' = \emptyset$. Set

$$\delta := 2^{-1} \Delta(\mathcal{O}') \tag{A.6}$$

and $\epsilon^* := (1/2)\beta^{-n} d(\mathcal{O}', D_\beta)$, so that $U_\beta^k|_{B(\mathcal{O}', \epsilon^*)}$ is continuous for all $0 \leqslant k \leqslant n-1$. For each $x \in \mathcal{O}'$, there exists $\epsilon_x \in (0, \epsilon^*)$ such that $\bigl|U_\beta^i(x) - U_\beta^i(y)\bigr| < \delta$ for all $y \in (x - \epsilon_x, x + \epsilon_x)$ and $0 \leqslant i \leqslant n-1$. Moreover, if $\epsilon := \min\{\epsilon_x : x \in \mathcal{O}'\}$ then

$$\bigl|U_\beta^k(x) - U_\beta^k(y)\bigr| < \delta \tag{A.7}$$

for all $x \in \mathcal{O}'$, $y \in B(x, \epsilon)$, and $0 \leqslant k \leqslant n-1$.

The sequence $\underline{y}$ is constructed recursively as follows.

Base step. Choose an arbitrary $y_{-1} \in \mathcal{O}'$, and mark y_{-1} as a bad point.

Recursive step. For some $t \in \mathbb{N}_0$, assume that $y_{-1}, y_0, \dots, y_{t-1}$ are defined.

If $d\bigl(U_\beta^t(a), \mathcal{O}'\bigr) < \epsilon$, choose $y_t \in \mathcal{O}'$ such that $d\bigl(U_\beta^t(a), \mathcal{O}'\bigr) = \bigl|U_\beta^t(a) - y_t\bigr|$, set $y_{t+i} := U_\beta^i(y_t)$ for each $1 \leqslant i \leqslant n-1$, and mark $y_t, y_{t+1}, \dots, y_{t+n-1}$ as good points.

If $d\bigl(U_\beta^t(a), \mathcal{O}'\bigr) \geqslant \epsilon$, let p_t be the number of bad points in $\{y_{-1}, y_0, \dots, y_{t-1}\}$, then set $y_t := U_\beta^{p_t}(y_{-1})$ and mark y_t as a bad point.

Note that the required (A.4) is immediate from the above construction. To prove (A.5), note that for each bad point y_i, $i \in \mathbb{N}_0$, we have $d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) \geqslant \epsilon \geqslant \epsilon d\bigl(U_\beta^i(a), y_i\bigr)$, and for each good point y_i, we obtain $d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) = d\bigl(U_\beta^i(a), y_i\bigr)$ by (A.6), (A.7), and our construction. Therefore (A.5) holds by choosing $C_\mu := \max\{1, \, \epsilon^{-\alpha}\}$, so the Claim is proved for Case 1.

Case 2. Assume that $\mathcal{O}'$ is the orbit of 1 under U_β. By Remark 3.11, β is a simple beta-number. By Proposition 3.13 (i), $D_\beta \cap \mathcal{O}'(1) = U_\beta^{-1}(1) \cap \mathcal{O}' = \bigl\{U_\beta^{n-1}(1)\bigr\}$. Set

$$\delta' := \min\bigl\{2^{-1}\Delta(\mathcal{O}'(1)),\, 2^{-1}d(0, \mathcal{O}'(1))\bigr\}. \tag{A.8}$$

For each $0 \leqslant i \leqslant n-1$ and $z_i := U_\beta^i(1)$, we claim that there exists $\epsilon_i > 0$ satisfying the following properties:

(1) $\bigl|U_\beta^j(z_i) - U_\beta^j(y)\bigr| < \delta'$ for all $y \in (z_i - \epsilon_i, z_i]$, $0 \leqslant j \leqslant n-1$.

(2) $\bigl|U_\beta^j(z_i) - U_\beta^j(y)\bigr| < \delta'$ for all $y \in (z_i, z_i + \epsilon_i)$, $0 \leqslant j \leqslant n-i-1$.

(3) $\bigl|U_\beta^j(y)\bigr| < \delta'$ for all $y \in (z_i, z_i + \epsilon_i)$, $n-i \leqslant j \leqslant n-1$.

Indeed, by Lemma 3.4 (i), $\lim_{y \nearrow x} U_\beta^j(y) = U_\beta^j(x)^-$ for each $x \in (0,1]$ and each $j \in \mathbb{N}$. Thus, there exists $\epsilon_{i,1} > 0$ satisfying property (1).

Now let $A_i := \emptyset$ when $i = n-1$ and $A_i := \bigl\{z_i, \dots, U_\beta^{n-i-2}(z_i)\bigr\}$ when $i < n-1$. Note that U_β is continuous on A_i for each $0 \leqslant i \leqslant n-1$ since $A_i \cap D_\beta = \emptyset$, so if $0 \leqslant j \leqslant n-i-1$ then U_β^j is continuous at z_i and $\lim_{y \searrow z_i} U_\beta^j(y) = U_\beta^j(z_i)^+$. Hence, there exists $\epsilon_{i,2} > 0$ satisfying property (2).

Since $U_\beta^{n-1}(1) \in D_\beta$, by (3.2), we get that $\lim_{y \searrow z_i} U_\beta^j(y) = 0$ for each $n-i \leqslant j \leqslant n-1$. Thus, there exists $\epsilon_{i,3} > 0$ satisfying property (3).

Defining $\epsilon_i := \min\{\epsilon_{i,1}, \epsilon_{i,2}, \epsilon_{i,3}\}$, we see that ϵ_i satisfies properties (1), (2), and (3).

Now define $\epsilon' := \min\{\epsilon_i : 0 \leqslant i \leqslant n-1\}$, and construct $\underline{y}$ as in Case 1, except that ϵ and δ are replaced, respectively, by ϵ' and δ'. Then (A.4) holds immediately, while for each bad point y_i, $i \in \mathbb{N}_0$, we have $d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) \geqslant \epsilon' \geqslant \epsilon' d\bigl(U_\beta^i(a), y_i\bigr)$, and for each good point y_i, by (A.8) and properties (1), (2), and (3), either $d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) = d\bigl(U_\beta^i(a), y_i\bigr)$ or $d\bigl(U_\beta^i(a), \mathcal{O}'\bigr) \geqslant \delta' \geqslant \delta' d\bigl(U_\beta^i(a), y_i\bigr)$. Hence (A.5) holds if we take $C_\mu := \min\{(\epsilon')^{-\alpha}, (\delta')^{-\alpha}, 1\}$, so the Claim is proved for Case 2. $\square$

Having established Lemma A.2, we are now ready to prove Theorem A.1.

Proof of Theorem A.1. If $\phi \in \mathscr{P}^\alpha(\beta)$, choose $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ supported on a periodic orbit $\mathcal{O}'$ of U_β, and for each $t > 0$ define

$$\phi_t := \phi - t d(\cdot, \mathcal{O}')^\alpha \in C^{0,\alpha}(I).$$

Clearly ϕ_t belong to $C^{0,\alpha}(I)$, and converge to ϕ as $t \to 0$. By Lemma A.2, if $t > 0$, $\nu \in \mathcal{M}(I, U_\beta) \smallsetminus \{\mu\}$, and $\psi \in C^{0,\alpha}(I)$ with $\|\psi\|_{\alpha,I} < t/C_\mu$, then

$$\begin{aligned}
\int_I (\phi_t + \psi)\,\mathrm{d}\nu &= \int_I \phi\,\mathrm{d}\nu + \int_I \psi\,\mathrm{d}\nu - t\int_I d(\cdot, \mathcal{O}')^\alpha\,\mathrm{d}\nu \\
&\leqslant \int_I \phi\,\mathrm{d}\mu + \int_I \psi\,\mathrm{d}\mu + (C_\mu|\psi|_{\alpha,I} - t)\int_I d(\cdot, \mathcal{O}')^\alpha\,\mathrm{d}\nu \\
&= \int_I (\phi_t + \psi)\,\mathrm{d}\mu + (C_\mu|\psi|_{\alpha,I} - t)\int_I d(\cdot, \mathcal{O}')^\alpha\,\mathrm{d}\nu \\
&< \int_I (\phi_t + \psi)\,\mathrm{d}\mu,
\end{aligned}$$

so $\mathcal{M}_{\max}\bigl(U_\beta, \phi_t + \psi\bigr) = \{\mu\}$, and consequently, $\phi_t \in \mathrm{Lock}^\alpha(\beta)$. Hence, $\mathrm{Lock}^\alpha(\beta)$ is dense in $\mathscr{P}^\alpha(\beta)$. Moreover, from their definitions, $\mathrm{Lock}^\alpha(\beta)$ is an open subset of $\mathscr{P}^\alpha(\beta)$, therefore the theorem is established. $\square$

Remark A.3. Using the proof above, we provide a counter-example for the TPO Conjecture for T_β when β is a simple beta-number. Let $\beta > 1$ be a simple beta-number and $\alpha \in (0, 1]$, define $\phi \coloneqq -d\big(\cdot, \mathcal{O}'_\beta(1)\big)^\alpha$. According to the proof above, ϕ has the locking property in $C^{0,\alpha}(I)$ with the unique U_β-maximizing measure equal to $\mu_{\mathcal{O}'_\beta(1)}$. Since $\mu_{\mathcal{O}'_\beta(1)}$ is not a T_β-invariant measure, by Proposition 4.4 (ii) and $\mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$ (Proposition 3.13 (iii)), any function sufficiently close to ϕ has no T_β-maximizing measure, and consequently does not have the locking property in $C^{0,\alpha}(I)$.

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