JOINT TYPICAL PERIODIC OPTIMIZATION

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Dedicated to the memory of Gonzalo Contreras

Abstract. We prove a generalised Yuan–Hunt–Mañé Conjecture: if $\mathcal{F}$ is the Banach space of α-Hölder functions, and $\mathcal{T}$ is either a space of Lipschitz expanding maps, or of Anosov diffeomorphisms, or the family of beta-transformations on the interval, there is an open dense subset of $\mathcal{T} \times \mathcal{F}$ consisting of map-function pairs whose maximizing invariant measure is unique and supported on a periodic orbit.

Contents

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Introduction

In this article we prove a generalised Yuan–Hunt–Mañé Conjecture, in the spirit of related work of Hunt & Ott [HO96a, HO96b], Yang, Hunt & Ott [YHO00], and Mañé [Man96, Man97].

Contreras [Co16] proved the Yuan–Hunt Conjecture [YH99] that ground states are generically periodic, more precisely:

Theorem. *If X is a compact metric space, and $T : X \to X$ is an open Lipschitz expanding map, then there exists an open dense subset U of the space $\mathrm{Lip}(X)$ of real-valued Lipschitz functions, such that if $\phi \in U$ then the unique (T, ϕ)-maximizing measure is supported on a periodic orbit.*

In this article we go further, allowing both the function ϕ and the dynamical system T to vary. If $\mathcal{E}(X)$ denotes the space of all open Lipschitz expanding maps on X, we prove:

Theorem. *If X is a compact locally connected metric space, then there exists an open dense subset U of $\mathcal{E}(X) \times \mathrm{Lip}(X)$ such that if $(T, \phi) \in U$ then the unique (T, ϕ)-maximizing measure is supported on a periodic orbit.*

The above theorem applies, in particular, to the case where $\mathcal{E}(X)$ is the space of all Lipschitz expanding maps on a compact Riemannian manifold X. For such X we also prove an analogous theorem for the space of all Anosov diffeomorphisms on X. Going beyond the uniformly hyperbolic setting, for $\beta > 1$ we consider beta-transformations $T_\beta(x) = \beta x \pmod 1$ on the unit interval I, and prove:

Theorem. *There is an open dense subset U of $(1, +\infty) \times \mathrm{Lip}(I)$ such that if $(\beta, \phi) \in U$ then the unique (T_β, ϕ)-maximizing measure is supported on a periodic orbit.*

The proof of these results relies on a novel perturbation technique, that we refer to as *joint perturbation.* In the case of beta-transformations, two distinct perturbation arguments are employed at different stages of the proof, one revolving around *simple* beta-numbers, the other around *non-simple* beta-numbers.

1. Statement of results

1.1. **Background.** An important strand in dynamical systems theory is the understanding of those properties that are in some sense typical, or robust under perturbation of the system. Investigations along these lines include classical works of Oxtoby & Ulam [OU41], Halmos [Ha44], and Rokhlin [Rok48], on genericity in the context of ergodicity, mixing, and weak mixing, and the considerable literature on structural stability and generic properties of diffeomorphisms (see e.g. [BC04, BD96, BDP03, GPS94, Man82, Man88, MV93, New79, Pu67, PS00, Rob71, SW00]), and on typical properties of one-dimensional systems (see e.g. [AMM03, GrSw97, GrSw98, Ly97, Ly98, Ly02]); for notions of typicality in dynamical systems, see e.g. the survey by Hunt & Kaloshin [HK10].

Of particular importance, in view of physical applications to predictability and stability, are situations where the solution of a variational problem is typically a periodic orbit, and the selection of this orbit is robust under all perturbations. One such problem is described by specifying a dynamical system $T\colon X \to X$, and a (potential) function $\phi\colon X \to \mathbb{R}$, and investigating those T-invariant probability measures that maximize (or minimize) the corresponding ergodic average. First considered in the 1990s, numerical work on this *ergodic optimization* problem suggested that typically there is a simple solution: the maximizing measure seems to often be periodic, i.e., supported on a single periodic orbit. More precisely, if ϕ is chosen from a space of suitably regular (e.g. smooth, or Hölder continuous) functions, and T is chosen from a space of suitably chaotic maps, it appears to be the case that for a typical pair (T, ϕ), there exists a T-periodic orbit $\mathcal{O} = \mathcal{O}_{T,\phi}$ such that the ϕ-average $\frac{1}{\operatorname{card}\mathcal{O}} \sum_{x\in\mathcal{O}} \phi(x)$ is strictly larger than the ϕ-average $\int \phi \, \mathrm{d}\mu$ with respect to any T-invariant probability measure μ not supported on $\mathcal{O}$.

In 1996, Hunt & Ott [HO96a, HO96b] conjectured, on the basis of numerical experiments for various parametrised families of chaotic maps T and smooth functions ϕ, that the set of parameter values for which maximizing measures are periodic has full Lebesgue measure; this conjecture has been confirmed in a number of specific settings (see [BZ16, DLZ24, GSZ25][1]). In 1999, Yuan & Hunt [YH99] conjectured a topological analogue, that for any uniformly hyperbolic dynamical system T (specifically, an expanding map or an Axiom A diffeomorphism), the maximizing measure is periodic for an open and dense subset of Lipschitz functions ϕ. This conjecture generated a sustained period of work over the following years (see [Bou01, Bou08, CLT01, Mo08, QS12]), and was eventually proved by Contreras [Co16] for expanding maps[2], and by Huang, Lian, Ma, Xu & Zhang [HLMXZ19] for the remaining uniformly hyperbolic cases; going beyond the setting of uniform hyperbolicity, Li & Zhang [LZ25] proved the analogous result for expanding Thurston maps. Progress towards the Hunt–Ott Conjecture and the Yuan–Hunt Conjecture, and broader advances in the field of ergodic optimization (see e.g. [Boc18, Je06, Je19] for further details), prompted the Typical Periodic Optimization (TPO) Conjecture (cf. [Je19]), that if T is suitably hyperbolic, and $\mathcal{F}$ is a Banach space consisting of suitably regular continuous functions, then the set of functions with a periodic maximizing measure will contain an open dense subset of $\mathcal{F}$.

A parallel conjecture of Mañé ([Man96, Problem III], [Man97, p. 143]) concerning the minimizing measures of Lagrangian flows, as introduced by Mather [Mat89, Mat91], is that for a generic Lagrangian there is a single minimizing measure, and it is supported on a periodic orbit. Mañé's Conjecture has been influential in Aubry–Mather theory [CFR15], and in the study of the Hamilton–Jacobi equation [DFIZ16, FS04], and has stimulated the cross-fertilisation of ideas with ergodic optimization (notably, [CLT01] is influenced by Mañé's work on Aubry–Mather theory, while [Co24] employs ergodic optimization techniques to tackle Mañé's Conjecture). In the Mañé–Mather formulation, the Lagrangian whose action is to be optimized determines the flow itself, and this binding of potential function to dynamics has inspired workers in ergodic optimization, beginning with [CLT01], to study generic properties of measures that optimize Lyapunov exponents (see also [GaSh24, JM08, MT13, ST20]).

1.2. **Joint typical periodic optimization.** In the present article we undertake a programme investigating the most general notion of topological typicality in ergodic optimization, by allowing the independent variation of both dynamical system and potential function. This approach, corresponding to the one suggested by Yang, Hunt & Ott [YHO00][3], and in [Je19, pp. 2610–2611], is most natural from the perspective of physical applications, allowing perturbation of all laws governing the optimization. Specifically, our interest is in the *joint typical periodic optimization (Joint TPO)* problem, allowing joint variation of pairs (T,ϕ) of map T and function ϕ in suitable *product spaces*, and seeking to establish that periodic optimization is a typical property with respect to the *topology of the product space*. More precisely, for any Banach space $\mathcal{F}$ consisting of suitable continuous real-valued functions on a compact metric space X, and a suitable space $\mathcal{T}$ of self-maps of X, we say that a pair $(T,\phi)\in\mathcal{T}\times\mathcal{F}$ has the *periodic optimization property* if there exists some T-periodic orbit $\mathcal{O}$, with corresponding T-invariant probability measure

$$\mu_{\mathcal{O}} \coloneqq \frac{1}{\operatorname{card}\mathcal{O}}\sum_{x\in\mathcal{O}}\delta_x, \tag{1.1}$$

such that

$$\int\phi\,\mathrm{d}\mu_{\mathcal{O}} > \int\phi\,\mathrm{d}\nu \qquad \text{for all } \nu\in\mathcal{M}(X,T)\smallsetminus\{\mu_{\mathcal{O}}\}, \tag{1.2}$$

where $\mathcal{M}(X,T)$ denotes the set of T-invariant Borel probability measures on X. In other words, if a measure $\mu\in\mathcal{M}(X,T)$ is called (T,ϕ)-*maximizing* whenever $\int\phi\,\mathrm{d}\mu=\sup_{\nu\in\mathcal{M}(X,T)}\int\phi\,\mathrm{d}\nu =: Q(T,\phi)$,

[1]In these works, "full Lebesgue measure" is expressed in terms of *prevalence*, see e.g. Hunt & Kaloshin [HK10].

[2]More precisely, the proof is for distance-expanding, Lipschitz, open mappings.

[3]The suggestion in [YHO00, p. 1951] concerns 'enlarging the meaning of typicality to be with respect to both variation of system parameters and performance function parameters'.

the periodic optimization property for (T, ϕ) means that there is a unique (T, ϕ)-maximizing measure, and it is of the form (1.1) for some T-periodic orbit $\mathcal{O}$.

For a fixed $T \in \mathcal{T}$, we say that $\{T\} \times \mathcal{F}$ has the *typical periodic optimization (TPO)* property if there is an open dense subset of $\mathcal{F}$ such that (T, ϕ) has the periodic optimization property for every element ϕ of the subset. Whenever the function space $\mathcal{F}$ is clear from the context, we simply say that *T has TPO*.

We say that the pair $(\mathcal{T}, \mathcal{F})$ (or indeed the space $\mathcal{T} \times \mathcal{F}$, equipped with the product topology) has the *joint typical periodic optimization (Joint TPO)* property if there is an open dense set of pairs (T, ϕ) in the product space $\mathcal{T} \times \mathcal{F}$ that have the periodic optimization property.[4] In order to distinguish between these notions, it will sometimes be convenient to refer to TPO as *Individual TPO*. As we explain below, typical properties in the product space $\mathcal{T} \times \mathcal{F}$ neither imply, nor are implied by, typical properties in typical fibres $\{T\} \times \mathcal{F}$, so a Joint TPO result is not a straightforward consequence of an Individual TPO result, and an Individual TPO result is not a straightforward consequence of a Joint TPO result.[5]

In the initial part of this article we consider joint typical periodic optimization in settings for which Individual TPO is known to hold, where the members of $\mathcal{T}$ are uniformly hyperbolic (expanding maps or Anosov diffeomorphisms)[6], and $\mathcal{F}$ is either the Hölder space $C^{0,\alpha}(X)$, or (where appropriate) the C^1 function space; in the latter part of the article we address Joint TPO in the non-uniformly hyperbolic setting of beta-transformations.

We first consider Lipschitz expanding maps, which in particular includes the class of all smooth expanding maps on arbitrary compact connected Riemannian manifolds, as studied notably by Shub [Sh69, Sh70] (see also e.g. [ES68, FJ78, GoRo23, Gr81, Hirs70] and [URM22, Chapter 6]). More specifically, if X is a compact locally connected metric space, and $\mathcal{E}(X)$ is the space of Lipschitz open mappings on X that are distance-expanding, we prove:

Theorem A (Joint TPO for expanding maps). *Suppose X is a compact locally connected metric space, and $\alpha \in (0, 1]$. There is an open dense subset of pairs $(T, \phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X)$ with the periodic optimization property.*

Note that typicality in $\mathcal{E}(X) \times C^{0,\alpha}(X)$ is in general not a straightforward consequence of typicality in each $\{T\} \times C^{0,\alpha}(X)$: indeed purely topological arguments do not even imply a weaker joint *generic* periodic optimization[7] result as an automatic consequence of Individual TPO in each fibre.[8] We next consider Anosov diffeomorphisms:

Theorem B (Joint TPO for Anosov diffeomorphisms). *Let M be a smooth compact Riemannian manifold, with distance function induced by the Riemannian metric, and let $\mathcal{A}(M)$ be the space of C^1 Anosov diffeomorphisms on M, equipped with the C^1 topology. For all $\alpha \in (0, 1]$, there is an open dense subset of pairs $(T, \phi) \in \mathcal{A}(M) \times C^{0,\alpha}(M)$ with the periodic optimization property.*

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[4]Although conceptually related to Mañé's Lagrangian framework (where flow and Lagrangian are rigidly coupled variables), and the Lyapunov optimization of [CLT01], the Joint TPO results in this article reveal a stronger stability phenomenon: periodic optimization, when achieved at a given system-potential pair in a certain open dense set, persists under arbitrarily small *independent* perturbations of both components.

[5]The situation is more subtle if we are content with *generic* statements, but even here there is little connection between joint results and fibrewise results; a notable exception is Theorem F, which is a straightforward consequence of a Joint TPO result.

[6]It is possible to prove Joint TPO results for more general hyperbolic dynamical systems, but such proofs are beyond the scope of the present work and will be presented elsewhere.

[7]By *joint generic periodic optimization* we mean that a *residual* subset of the product space $\mathcal{E}(X) \times C^{0,\alpha}(X)$ has the periodic optimization property.

[8]To justify this assertion, note that provided $\mathcal{E}(X)$ contains an embedded curve, there exists a subset $\mathcal{L} \subseteq \mathcal{E}(X) \times C^{0,\alpha}(X)$ that is not residual in $\mathcal{E}(X) \times C^{0,\alpha}(X)$, yet for each $T \in \mathcal{E}(X)$, the set $\{\phi \in C^{0,\alpha}(X) : (T, \phi) \in \mathcal{L}\}$ contains an open dense subset of $C^{0,\alpha}(X)$; this is a consequence of the fact (see [Ox71, Theorem 15.5], and cf. [Ke95, p. 54]) that there exists a non-meagre subset of $[0, 1]^2$ such that no three of its elements are collinear (cf. also footnote 9).

Further is this direction, in [HHJL25], we are able to apply our methods towards joint TPO for more general cases and more kinds of dynamics. In particular, based on the important works [Ly97], [MSS83], and [Ly83] in complex dynamics, we are able to establish joint TPO for $\mathcal{T}$ being the space of real quadratic polynomials on the Riemann sphere, and based on the important work [KSS07] in real one-dimensional dynamics, we are able to establish joint TPO for $\mathcal{T}$ being the space of C^r ($r \geqslant 1$) maps on a compact interval or a circle. One remarkable thing is our joint TPO programme becomes an application of the fundamental works [Ly97] and [KSS07] about density of hyperbolicity, which were hardly used before.

A more delicate context for joint typical periodic optimization results, beyond the uniformly hyperbolic setting, arises if we take the space $\mathcal{T}$ to be a classical one-parameter family of maps of the unit interval $I = [0,1]$, the *beta-transformations* $T_\beta(x) = \beta x \pmod 1$, $\beta > 1$. Unlike for Lipschitz open distance-expanding maps, not every beta-transformation has TPO: for example if $\beta = 2$ then the function ϕ given by $\phi(x) \coloneqq x$ has no maximizing measure, and it is readily checked that there is an open neighbourhood of ϕ in $C^{0,1}(I)$ consisting of functions with no maximizing measure. Moreover, it can be shown that there is a dense collection of parameters $\beta \in (1, +\infty)$ for which T_β does not have TPO (cf. Remark 5.21 and [Pa60, Theorem 5]). Perhaps surprisingly, we are nevertheless able to establish that Joint TPO *does* hold for the class of beta-transformations, where $\mathcal{T} = (1, +\infty)$ is equipped with its usual topology, and $\mathcal{T} \times \mathcal{F} = (1, +\infty) \times C^{0,\alpha}(I)$ is equipped with the product topology:

Theorem C (Joint TPO for beta-transformations). *Fix $\alpha \in (0,1]$. There is an open dense subset of pairs $(\beta, \phi) \in (1, +\infty) \times C^{0,\alpha}(I)$ such that (T_β, ϕ) has the periodic optimization property.*

From a technical perspective, the proof of Theorem C is more challenging than for Theorems A and B: the strategy of proof is outlined in Subsection 1.3 below, and is mediated by symbolic dynamics, exploiting amongst other things the fact that the orbit structure enjoys certain monotonicity properties as β varies. Despite the fact that not every T_β has TPO, we can nevertheless prove Individual TPO results for *most* parameters β, in the sense of the following two theorems:

Theorem D (Individual TPO for generic parameters β). *Fix $\alpha \in (0,1]$. For a residual set of values $\beta > 1$, the beta-transformation T_β has TPO (i.e., there is an open dense subset $V_\beta \subseteq C^{0,\alpha}(I)$ such that if $\phi \in V_\beta$ then (T_β, ϕ) has the periodic optimization property).*

Theorem E (Individual TPO for almost every parameter β). *Fix $\alpha \in (0,1]$. For Lebesgue almost every $\beta > 1$, the beta-transformation T_β has TPO (i.e., there is an open dense subset $V_\beta \subseteq C^{0,\alpha}(I)$ such that if $\phi \in V_\beta$ then (T_β, ϕ) has the periodic optimization property).*

Here again, a Joint TPO result is not a mere consequence of Individual TPO holding in most fibres[9]. Conversely, the joint typical periodic optimization of Theorem C does not automatically imply Theorem D: indeed the non-separability of $C^{0,\alpha}(I)$ means there exists an open dense subset[10] $\mathcal{L}$ of $(1,+\infty) \times C^{0,\alpha}(I)$ such that for all $\beta > 1$, the fibre $\left\{\phi \in C^{0,\alpha}(I) : (\beta, \phi) \in \mathcal{L}\right\}$ is not even dense in $C^{0,\alpha}(I)$. However, Theorem C does readily imply (cf. [Ke95, Lemma 8.42]) the following Theorem F, which can be considered an analogue of Theorem D, with the roles of maps and (potential) functions interchanged.

Theorem F (Individual TPO for generic potentials). *Fix $\alpha \in (0,1]$. There is a residual subset $R \subseteq C^{0,\alpha}(I)$ such that for all $\phi \in R$, there is an open and dense set of parameters $B_\phi \subseteq (1, +\infty)$ such that if $\beta \in B_\phi$ then (T_β, ϕ) has the periodic optimization property.*

[9] Specifically, the existence of a non-meagre subset $A \subseteq [0,1]^2$ such that no three of its points are collinear (cf. [Ox71, Theorem 15.5], [Ke95, p. 54]) means that if $\phi \in C^{0,\alpha}(I)$ is not identically zero then the embedding $\iota \colon [0,1]^2 \to (1,+\infty) \times C^{0,\alpha}(I)$ given by $\iota(x,y) = (x+2, y \cdot \phi)$ is such that $\iota(A)$ is non-meagre, yet the set $\left\{\psi \in C^{0,\alpha}(I) : (\beta, \psi) \in \iota(A)\right\}$ contains at most two points, hence is nowhere dense, for every $\beta \in (1, +\infty)$.

[10] Specifically, for each $x \in I$, set $f_x \coloneqq d(\cdot, x)^\alpha \in C^{0,\alpha}(I)$, then let $\psi \colon I \to (1, +\infty)$ be a bijection, and define $\mathcal{L}$ to be the complement of $\bigcup_{x \in I} \{\psi(x)\} \times \overline{B}(f_x, 1/3)$.

Theorems A, B, C, D, E, and F of course all imply corresponding *typical uniqueness* results: an open dense subset for which the maximizing measure is *unique*. Such results are analogous to Mañé's theorem [Man96, Theorem A] on uniqueness of the minimizing measure for generic Lagrangians, and (individual) generic uniqueness results in ergodic optimization (see e.g. [CLT01, Theorem 6] and [Je06, Theorem 3.2]), though these latter results assert uniqueness only on a residual subset, rather than the open dense subset implied by our theorems. One consequence of this *joint typical uniqueness* is a *large deviation principle* as $t \to +\infty$ for families of $(T, t\phi)$-equilibrium states, for an open dense subset of pairs $(T, \phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X)$ (see Remark 2.11).

1.3. **Heuristic overview of the proofs.** Very broadly, the proof of the main theorems can be split into the following steps:

Case $\mathcal{H}$: *Uniform hyperbolicity. Proof strategy for Theorems A and B.*

- **Step $\mathcal{H}$.1:** *(Exploit structural stability and shadowing).* For open expanding maps, our local connectedness hypothesis ensures suitable control of the surjectivity radius (Proposition 2.2), yielding the Locally Connected Shadowing Lemma (Lemma 2.5), guaranteeing a weak form of structural stability for perturbed systems, which is needed in Step $\mathcal{H}$.3 below. For Anosov diffeomorphisms we directly use structural stability.
- **Step $\mathcal{H}$.2:** *(A Mañé Lemma with semi-norm control).* A crucial second step is an enhanced *Mañé cohomology lemma* (Theorem 2.6 in the case of open expanding maps): there is a uniform $L = L(T, \alpha) > 0$ such that Hölder functions ϕ admit a Hölder sub-action u, with $|u|_\alpha \leqslant L|\phi|_\alpha$. The need for this additional control is a distinctive feature of Joint TPO (as compared to Individual TPO).
- **Step $\mathcal{H}$.3:** *(Joint perturbation).* The joint perturbation theorems (Theorems 2.7 and 2.12) are the key technical results towards Theorems A and B. Fixing a map T_0, and letting $\mathcal{O}_0$ be any of its periodic orbits, we then show that for all T sufficiently close to T_0, there is a T-periodic orbit $\mathcal{O}$ such that for suitable Hölder functions whose unique T_0-maximizing measure is supported by $\mathcal{O}_0$, there is a nearby function whose unique T-maximizing measure is supported by $\mathcal{O}$.
- **Step $\mathcal{H}$.4:** *(Joint periodic locking).* For open expanding maps we define the *joint periodic locking set* $\mathfrak{L}^\alpha(X) = \{(T, \phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X) : \phi \in \mathrm{Lock}^\alpha(T)\}$, where each individual locking set $\mathrm{Lock}^\alpha(T)$ consists of those α-Hölder functions whose T-maximizing measure is unique, periodic, and stably maximizing under perturbations. The Joint Perturbation Theorem (Theorem 2.7) is used to show that $\mathfrak{L}^\alpha(X)$ is *open*, and Contreras' Individual TPO Theorem is used to show that $\mathfrak{L}^\alpha(X)$ is *dense*. For Anosov diffeomorphisms an analogous argument is used, together with Theorem 2.12.

Case β: *Beta-transformations. Proof strategy for Theorems C, D, E, and F.*

- **Step β.1:** *(Compactification).* A necessary first step is to develop a theory of ergodic optimization for the T_β, accommodating the fact that these maps are not continuous, and that for certain β the set $\mathcal{M}(I, T_\beta)$ of T_β-invariant probability measures is not compact. The tool here is the *upper beta-transformation* $U_\beta \colon I \to I$, defined by $U_\beta(0) := 0$ and $U_\beta(x) := \beta x - \lfloor \beta x \rfloor'$ for $x \in (0, 1]$, where $\lfloor \beta x \rfloor' := \max\{n \in \mathbb{Z} : n < \beta x\}$. We show that the set $\mathcal{M}(I, U_\beta)$ of U_β-invariant probability measures is weak* compact, and equal to the closure of $\mathcal{M}(I, T_\beta)$, so a (U_β, ϕ)-maximizing measure exists whenever ϕ is continuous.
- **Step β.2:** *(Structural monotonicity).* We leverage a fundamental property of beta-transformations: the structure of their corresponding symbolic dynamical systems (beta-shifts) has a certain monotonicity as β varies (cf. Lemma 3.16). This property forms the basis for two steps: the first is simple beta-number perturbation (Step β.5), the second is the Joint Perturbation Theorem for beta-transformations (Theorem 6.4, cf. Step β.9). Corollary 6.2,

as employed in Theorem 6.4, represents an analogue of the Locally Connected Shadowing Lemma (Lemma 2.5) of Step $\mathcal{H}.1$, and although the family of beta-transformations does not enjoy structural stability[11], this weaker shadowing property is nevertheless suitable for exploitation in the proof of Theorem 6.4.

Step β.3: *(Beta-numbers).* For certain β, the *critical orbit* (i.e., the orbit of 1 under U_β) is finite: in this postcritically-finite case β is called a *beta-number* (following [Pa60]). A result underpinning the Joint TPO result (Theorem C) is that, for *upper* beta-transformations, Individual TPO holds for *beta-numbers* (note that the same is *not* true for T_β, cf. Remark 5.21):

Theorem G (Individual TPO for beta-numbers). *Fix $\alpha \in (0,1]$. If $\beta > 1$ is a beta-number, then U_β has TPO (i.e., there is an open dense subset $V_\beta \subseteq C^{0,\alpha}(I)$ such that if $\phi \in V_\beta$ then (U_β, ϕ) has the periodic optimization property).*

Step β.4: *(Emergent parameters).* To prove Theorems D and E we introduce the notion of *emergent* parameters β: for an emergent $\beta > 1$, the symbolic dynamics for the critical orbit is essentially different from that witnessed in beta-shifts with parameter strictly smaller than β, so is considered to have newly *emerged* at this particular β. For the complementary parameter set we prove:

Theorem H (Individual TPO for non-emergent parameters). *Fix $\alpha \in (0,1]$. If $\beta > 1$ is non-emergent, then both T_β and U_β have TPO (i.e., there is an open dense subset $V_\beta \subseteq C^{0,\alpha}(I)$ such that if $\phi \in V_\beta$ then both (T_β, ϕ) and (U_β, ϕ) have the periodic optimization property).*

The set of emergent parameters can be shown to be both topologically meagre and of zero Lebesgue measure, so Theorems D and E follow from Theorem H.

Step β.5: *(Simple beta-number perturbation).* A key part of our overall strategy, employed to prove Theorems D, E, G, and H, is a perturbation argument in the space of parameters $\beta > 1$. This exploits the fact that the dynamics for a given β is approximable by sub-systems corresponding to so-called *simple* beta-numbers in $(1, \beta)$.

Step β.6: *(A Mañé lemma for beta-transformations).* To prove Theorems D, E, G, and H, the simple beta-number perturbation works in conjunction with a new Mañé cohomology lemma for beta-transformations (Theorem 4.9). Results of this kind (cf. Step $\mathcal{H}.2$ above) have been known since [Bou00, CLT01], and assert that cohomology classes of suitably regular functions contain versions (so-called *revealed versions*, cf. [Je19]) for which the maximizing measures are readily apparent; in favourable settings the revealed version inherits the modulus of continuity of the original function (see e.g. [Bou00, Bou01, BJ02, CLT01, LZ25]). Our Mañé lemma is for Hölder functions, and asserts the existence of *two* revealed versions, both enjoying one-sided continuity (one is left-continuous, the other right-continuous): the critical orbit introduces discontinuities, but away from this orbit both versions are locally Hölder. In particular, when β is a beta-number, the revealed versions are piecewise Hölder (with finitely many pieces): this analogue of Step $\mathcal{H}.2$ is a key ingredient for the proof of Theorem C. The proof of the beta-transformations Mañé lemma involves an operator fixed point that is a Borel measurable function with one-sided limits everywhere, and locally Hölder away from the critical orbit: this fixed point can then be *regularised*, yielding both a left-continuous and a right-continuous *sub-action*,

[11] As noted in Step $\mathcal{H}.1$, the structural stability of Anosov diffeomorphisms is exploited directly during the joint perturbation step. For open expanding maps, although it is possible to prove structural stability (a result that does not appear to be in the literature in the generality of open distance-expanding maps on compact locally connected spaces), it is more convenient to instead establish the Locally Connected Shadowing Lemma (Lemma 2.5).

allowing the definition of left-continuous and right-continuous revealed versions.[12] The one-sided continuity of these revealed versions allows us to deduce a Revelation Theorem (Theorem 4.12), asserting that any maximizing measure is supported within the union of the sets of maxima of these revealed versions; this serves as an important part of the following Step β.7.

Step β.7: *(Critical-regular analysis).* The other key ingredient for proving Theorems D, E, G, and H is the identification of two fundamental subsets of $C^{0,\alpha}(I)$, the *critical set* $\mathrm{Crit}^\alpha(\beta)$, and the *regular set* $\mathcal{R}^\alpha(\beta)$. The first of these sets consists of functions for which the critical orbit is maximizing, and the second consists of those functions enjoying good restrictions to various Cantor subsets on which T_β acts as an open expanding map: by exploiting Contreras' Individual TPO theorem [Co16] for such maps, we are able to prove the Dense Regular Functions Theorem (Theorem 5.16), asserting that $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$. This, together with the Revelation Theorem (Theorem 4.12), yields a proof of Theorems G and H.[13] It is an open problem to determine whether U_β has TPO for emergent parameters β: here a finer analysis leads to a Structural Theorem (Theorem 5.22), identifying $\mathrm{Crit}^\alpha(\beta)$ as a potential obstacle to Individual TPO.

Step β.8: *(Joint TPO for beta-transformations).* Having established the above Individual TPO theorems for beta-transformations, the conversion of this into the Joint TPO result Theorem C is patterned, to some extent, on the approach used in Steps $\mathcal{H}$.1–$\mathcal{H}$.4 to prove Theorems A and B. For beta-transformations, however, the technical analysis required is significantly more delicate, and the fine structure of this particular family of maps must be exploited. The strategy consists of proving a more general Theorem C′, asserting that Joint TPO holds for upper beta-transformations as well as for beta-transformations: in each case we show that the joint periodic locking set contains an open dense subset of $(1, +\infty) \times C^{0,\alpha}(I)$.

Step β.9: *(Joint perturbation for beta-transformations).* The proof of Theorem C′ relies on the Joint Perturbation Theorem for beta-transformations (Theorem 6.4), the most technically challenging result of the article. Although similar in spirit to the analogous joint perturbation theorems for expanding maps (Theorem 2.7) and Anosov diffeomorphisms (Theorem 2.12), the additional difficulties encountered in proving Theorem 6.4 relate to the non-persistence of certain periodic orbits, the discontinuous variation of some points under perturbation of the parameter, the weaker Mañé lemma (Theorem 4.9), and the consequent difficulties in controlling ergodic averages.[14] The proof of Theorem 6.4 centres on two key features: on the one hand *non-simple* beta-numbers β (these are shown to be dense in parameter space, and their specific properties are systematically exploited throughout the proof), and on the other hand a perturbation using the beta-transformations Mañé lemma (Theorem 4.9). Once Theorem C′ is proved, Theorem F follows readily.

1.4. **Organisation of the article.** Some key notation and terminology used throughout the article is fixed in Subsection 1.5 below, including the notion of periodic locking set $\mathrm{Lock}^\alpha(T)$, consisting of those α-Hölder functions whose maximizing measure is unique, periodic, and stably maximizing under perturbations. In Section 2 we introduce the space $\mathcal{E}(X)$ of open Lipschitz distance-expanding

[12] Since beta-transformations are neither continuous nor open, but on the other hand they are transitive, the approach to proving this Mañé lemma is rather different from, and more technical than, the one used in Step $\mathcal{H}$.2; see Remark 4.11 for more details.

[13] Note that, by contrast with [Co16, HLMXZ19, LZ25], the method for proving our Individual TPO theorems does not make explicit use of shadowing (indeed the shadowing property does not hold for beta-transformations, cf. [BGS25]).

[14] For further details see in particular the comments in footnotes 28 and 29 concerning Claim 3 and Subcase (ii) in the proof of Theorem 6.4.

maps on compact locally connected spaces X, and for all Hölder function spaces $C^{0,\alpha}(X)$ establish joint typical periodic optimization in the product space $\mathcal{E}(X) \times C^{0,\alpha}(X)$ (proved as the slightly stronger Theorem A′, asserting that $\{(T,\phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X) : \phi \in \mathrm{Lock}^{\alpha}(T)\}$ is itself an open dense subset). The special case when X is a compact Riemannian manifold follows (Theorem 2.10), and in this setting we also establish Joint TPO results for $\mathcal{T}$ the space of Anosov diffeomorphisms[15]: a slight strengthening of Theorem B is proved (as Theorem B′), as well as a Joint TPO theorem for C^1 function spaces (Theorem 2.13). Thereafter we focus on the more technically challenging setting of beta-transformations. Section 3, whose proofs are deferred to Appendix A, is preparatory in nature. Firstly, we establish various preliminary results about beta-transformations, beta-expansions, and the closely related beta-shifts, being particularly careful to distinguish the commonalities and differences between these systems; in so doing, we take the opportunity to clarify and correct some aspects of the published literature. Secondly, in order to develop ergodic optimization in this setting, we undertake an analysis of the set $\mathcal{M}(I,T_\beta)$ of T_β-invariant measures. We prove the existence of (U_β,ϕ)-maximizing measures for all values $\beta > 1$, and all continuous functions ϕ; we also introduce the notion of limit-maximizing measure, and establish the equivalence between maximizing measures for the upper beta-transformation and the beta-shift, and limit-maximizing measures for beta-transformations. In Section 4 we establish the important Mañé cohomology lemma for beta-transformations, and develop a revelation theorem as its consequence. Section 5 is devoted to proving our Individual TPO results for beta-transformations, and includes proofs of Theorems D, E, G, and H. To establish these results, the class of emergent parameters β is introduced, the critical subset $\mathrm{Crit}^{\alpha}(\beta)$ and regular subset $\mathcal{R}^{\alpha}(\beta)$ of $C^{0,\alpha}(I)$ are defined, and the Mañé lemma of Section 4 is a fundamental tool. In Section 6 we establish joint typical periodic optimization in the setting of beta-transformations: Theorem C is proved, via the slightly stronger Theorem C′, showing that both $\{(\beta,\phi) \in (1,+\infty) \times C^{0,\alpha}(I) : \phi \in \mathrm{Lock}^{\alpha}(T_\beta)\}$ and $\{(\beta,\phi) \in (1,+\infty) \times C^{0,\alpha}(I) : \phi \in \mathrm{Lock}^{\alpha}(U_\beta)\}$ contain open dense subsets of $(1,+\infty) \times C^{0,\alpha}(I)$. Theorem C′ is itself a consequence of the key joint perturbation result Theorem 6.4. Theorem F follows readily from Theorem C′. Appendix A is devoted to the proofs of the results stated in Section 3.

1.5. **Notation.** We follow the convention that $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For $x \in \mathbb{R}$, we define the floor function $\lfloor x \rfloor$ as the largest integer $\leqslant x$, and the strict floor function $\lfloor x \rfloor'$ as the largest integer $< x$. The cardinality of a set A is denoted by $\mathrm{card}\, A$.

The collection of all maps from a set X to a set Y is denoted by Y^X. The constant zero function $\mathbb{0} \colon X \to \mathbb{R}$ maps each $x \in X$ to 0.

Let (X,d) be a metric space. For subsets $A, B \subseteq X$, we set $d(A,B) := \inf\{d(x,y) : x \in A,\, y \in B\}$, and $d(A,x) = d(x,A) := d(A,\{x\})$ for each $x \in X$. For each subset $Y \subseteq X$, we denote the diameter of Y by $\mathrm{diam}(Y) := \sup\{d(x,y) : x,\, y \in Y\}$, for each $\varepsilon > 0$, denote the ε-neighbourhood of Y (in X) by $B(Y,\varepsilon) := \{x \in X : d(x,Y) < \varepsilon\}$, and the closure of $B(Y,\varepsilon)$ by $\overline{B}(Y,\varepsilon)$. For each $y \in X$ and each $\varepsilon > 0$, write $B(y,\varepsilon) := B(\{y\},\varepsilon)$.

For a finite non-empty set $F \subseteq X$, let $\Delta(F)$ denote its *minimum interpoint distance*, i.e., $\Delta(F) := \min\{d(x,y) : x,\, y \in F,\, x \neq y\}$ if $\mathrm{card}\, F \geqslant 2$ and $\Delta(F) := +\infty$ if $\mathrm{card}\, F = 1$.

Let $C(X)$ denote the space of continuous functions from X to $\mathbb{R}$, and $\mathcal{P}(X)$ the set of Borel probability measures on X. For $\phi \colon X \to \mathbb{R}$, we write $\|\phi\|_\infty = \|\phi\|_{\infty,X} := \sup\{|\phi(x)| : x \in X\}$.

Let (X,d) be a compact metric space and $\alpha \in (0,1]$. A function $\phi \colon X \to \mathbb{R}$ is called α-Hölder if

$$|\phi|_{\alpha,X} := \sup\{|\phi(x) - \phi(y)|/d(x,y)^{\alpha} : x,\, y \in X,\, x \neq y\} < +\infty.$$

Denote by $C^{0,\alpha}(X)$ the set of real-valued α-Hölder functions ϕ on X, equipped with the Hölder norm $\|\cdot\|_{\alpha,X}$ given by

$$\|\phi\|_{\alpha,X} := |\phi|_{\alpha,X} + \|\phi\|_{\infty,X},$$

[15]Note that the presentation for Anosov diffeomorphisms, in Subsection 2.5, is deliberately less detailed than for expanding maps and beta-transformations, in view of a forthcoming work on more general hyperbolic systems, and the similarity of certain arguments and estimates.

which makes $C^{0,\alpha}(X)$ a Banach space. It is often convenient to write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{\alpha,X}$: this will be done throughout Section 2, and is usually done in the case that $X = [0,1]$ (the exception being in Section 5, where X is chosen to be various different subsets of $[0,1]$).

For a Borel measurable map $T\colon X \to X$ on a compact metric space X, let $\mathcal{M}(X,T)$ denote the set of T-invariant Borel probability measures on X, and define the *ergodic supremum* of a bounded Borel measurable function $\psi\colon X \to \mathbb{R}$ to be

$$Q(T,\psi) \coloneqq \sup_{\nu\in\mathcal{M}(X,T)} \int \psi \,\mathrm{d}\nu. \tag{1.3}$$

Any measure $\mu \in \mathcal{M}(X,T)$ that attains the supremum in (1.3) is called a (T,ψ)-*maximizing measure*, and it will also be convenient to refer to μ as a ψ-*maximizing measure* (for the map T), and as a T-*maximizing measure* (for the function ψ). The set of (T,ψ)-maximizing measures is denoted by

$$\mathcal{M}_{\max}(T,\psi) \coloneqq \left\{\mu \in \mathcal{M}(X,T) : \int \psi \,\mathrm{d}\mu = Q(T,\psi)\right\}. \tag{1.4}$$

The orbit of a point $x \in X$ is called a *maximizing orbit* for (T,ψ) (or (T,ψ)-*maximizing*) if

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi\bigl(T^i(x)\bigr) = Q(T,\psi). \tag{1.5}$$

For a map $T\colon X \to X$ and a real-valued function $\phi\colon X \to \mathbb{R}$, define

$$S_n^T\phi(x) \coloneqq \sum_{i=0}^{n-1} \phi\bigl(T^i(x)\bigr) \qquad \text{for } x \in X, n \in \mathbb{N}.$$

Note that by definition $S_0^T\phi \equiv 0$. In the particular case where T is some beta-transformation T_β, we will write $S_n\phi = S_n^{T_\beta}\phi$ whenever there is no possibility of confusion.

We write $I \coloneqq [0,1]$. In this article, we equip every subset of I with the usual Euclidean metric, denoted by d.

For a non-empty set $\mathcal{A}$ equipped with the discrete topology, the sequence space $\mathcal{A}^{\mathbb{N}} = \{\{a_n\}_{n=1}^{+\infty} : a_n \in \mathcal{A} \text{ for all } n \in \mathbb{N}\}$ will be equipped with the product topology. For $t > 1$, the metric d_t on $\mathcal{A}^{\mathbb{N}}$ defined by $d_t(\{a_n\}_{n=1}^{+\infty}, \{b_n\}_{n=1}^{+\infty}) \coloneqq t^{-p}$, where p is the smallest positive integer with $a_p \neq b_p$, and $d_t(\{a_n\}_{n=1}^{+\infty}, \{b_n\}_{n=1}^{+\infty}) \coloneqq 0$ if $a_n = b_n$ for all $n \in \mathbb{N}$, generates the product topology on $\mathcal{A}^{\mathbb{N}}$.

Infinite sequences will be written as $A = a_1a_2\ldots = \{a_n\}_{n\in\mathbb{N}}$, and finite sequences as $B = b_1b_2\ldots b_k = \{b_n\}_{n=1}^k$. Denote $(b_1b_2\ldots b_k)^\infty \coloneqq b_1b_2\ldots b_kb_1b_2\ldots b_kb_1b_2\ldots$ and write $(b_1b_2\ldots b_k)^m$ for the first km terms of $(b_1b_2\ldots b_k)^\infty$, for $m \in \mathbb{N}$.

If $\mathcal{A} \subseteq \mathbb{R}$ is equipped with the order induced by $\mathbb{R}$, and $A, B \in \mathcal{A}^{\mathbb{N}}$, write $A \prec B$ when A has strictly smaller lexicographic order than B, i.e., $a_i = b_i$ for $1 \leqslant i \leqslant n-1$, and $a_n < b_n$, for some $n \in \mathbb{N}$. Write $A \preceq B$ to mean $A \prec B$ or $A = B$.

Define the (left) shift map

$$\sigma\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$$

by $\sigma(A) \coloneqq \{a_{n+1}\}_{n\in\mathbb{N}}$ for all $A = \{a_n\}_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

Let X be a topological space. For a map $T\colon X \to X$ and $x \in X$, denote the orbit of x by

$$\mathcal{O}^T(x) \coloneqq \{T^n(x) : n \in \mathbb{N}_0\}.$$

In this article, we write $\mathcal{O}_\beta(x) \coloneqq \mathcal{O}^{T_\beta}(x)$ and $\mathcal{O}_\beta^*(x) \coloneqq \mathcal{O}^{U_\beta}(x)$ whenever there is no possibility of confusion. If there exists $n \in \mathbb{N}$ with $T^n(x) = x$, then $\mathcal{O}(x)$ is called a periodic orbit and x a periodic point. We denote the set of periodic points of T by $\mathrm{Per}(T)$. There is a unique T-invariant Borel probability measure $\mu_{\mathcal{O}}$ supported on a periodic orbit $\mathcal{O}$, given by

$$\mu_{\mathcal{O}} \coloneqq \frac{1}{\operatorname{card}\mathcal{O}} \sum_{x\in\mathcal{O}} \delta_x.$$

For a topological space X, a point $a \in X$, and a function $f\colon X \to \mathbb{R}$, we write $\lim_{y\to a} f(y) = x^+$ if $\lim_{y\to a} f(y) = x$ and there exists a neighbourhood U of a such that $f|_{U\smallsetminus\{a\}} \geqslant x$. We define $\lim_{y\to a} f(y) = x^-$ similarly. We say that a sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ converges to a real number x^+ (written as $\lim_{n\to+\infty} x_n = x^+$) if $x_n \geqslant x$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = x$. We define $\lim_{n\to+\infty} x_n = x^-$ similarly. Moreover, for a real number a, and a function $g\colon \mathbb{R} \to \mathbb{R}$, we denote $\lim_{y\searrow a} g(y) \coloneqq \lim_{y\to a} g|_{(a,+\infty)}(y)$, that is, the right-hand limit of g at a. We denote the left-hand limit $\lim_{y\nearrow a} g(y) \coloneqq \lim_{y\to a} g|_{(-\infty,a)}(y)$ similarly. More generally, if $\mathbb{R}$ is replaced by a topological well-ordered set Y, the above notions and notation are defined similarly.

Fix a constant $\alpha \in (0,1]$, a compact metric space X, and a measurable map $T\colon X \to X$. We define subsets

$$\mathscr{P}^\alpha(T) \quad \text{and} \quad \mathrm{Lock}^\alpha(T)$$

of the set $C^{0,\alpha}(X)$ as follows: $\mathscr{P}^\alpha(T)$ is the set of those $\phi \in C^{0,\alpha}(X)$ with a ϕ-maximizing measure supported on a periodic orbit of T. If a function $\phi \in \mathscr{P}^\alpha(T)$ satisfies $\operatorname{card} \mathcal{M}_{\max}(T,\phi) = 1$ and $\mathcal{M}_{\max}(T,\phi) = \mathcal{M}_{\max}(T,\psi)$ for all $\psi \in C^{0,\alpha}(X)$ sufficiently close to ϕ in $C^{0,\alpha}(X)$, we say that ϕ has the *(periodic) locking property*[16] in $C^{0,\alpha}(X)$ (with respect to T). The set $\mathrm{Lock}^\alpha(T)$ is defined to consist of all $\phi \in \mathscr{P}^\alpha(T)$ satisfying the (periodic) locking property in $C^{0,\alpha}(X)$.

2. Joint TPO: Uniform hyperbolicity

In this section, we establish joint typical periodic optimization in the context of uniformly hyperbolic systems (the space of Lipschitz open distance-expanding maps, and the space of C^1 Anosov diffeomorphisms).

2.1. **Expanding maps.**

Recall the following notion (see e.g. [PU10, Chapter 4]):

Definition 2.1 (Distance-expanding map). For (X,d) a compact metric space, $T\colon X \to X$ is called a *distance-expanding map* if there exist constants $\lambda > 1$ and $\gamma > 0$ such that if $x, y \in X$ with $d(x,y) \in (0,\gamma)$, then $d(T(x),T(y)) > \lambda d(x,y)$.

We refer to (γ,λ) as a pair of *expanding constants* for the distance-expanding map T.

We shall be interested in the class of distance-expanding maps that are, moreover, Lipschitz and open (i.e., map open sets to open sets):

Notation. Suppose (X,d) is a compact metric space. A map $T\colon X \to X$ is *Lipschitz (continuous)* if its *Lipschitz constant*

$$\mathrm{LIP}(T) \coloneqq \sup_{x,\, y\in X,\, x\neq y} \frac{d(T(x),T(y))}{d(x,y)}$$

is finite. The set of Lipschitz continuous open distance-expanding maps $T\colon X \to X$ will be denoted by $\mathcal{E}(X)$, and equipped with the metric[17]

$$d_{\mathrm{Lip}}(T_1,T_2) \coloneqq \sup_{x\in X} d(T_1(x),T_2(x)) + \sup_{x,y\in X, x\neq y} \frac{|d(T_1(x),T_1(y)) - d(T_2(x),T_2(y))|}{d(x,y)} \tag{2.1}$$

for T_1, $T_2 \in \mathcal{E}(X)$. For each $T \in \mathcal{E}(X)$ and each $\alpha \in (0,1]$, define the *joint (periodic) locking set* $\mathfrak{L}^\alpha(X)$ by

$$\mathfrak{L}^\alpha(X) \coloneqq \big\{(T,\phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X) : \phi \in \mathrm{Lock}^\alpha(T)\big\}. \tag{2.2}$$

At this level of generality, open distance-expanding maps were investigated by Ruelle[18] [Ru78, §7.26], and systematically studied by Przytycki & Urbański [PU10, Chapter 3] (see also [URM22,

[16]The terminology follows [Boc19, BZ15], and is somewhat inspired by [Bou00, Je00].

[17]It is readily checked that d_{Lip} defines a complete metric.

[18]Emanating from his theory of Smale spaces, Ruelle [Ru78, p. 149] notes that this theory of open distance-expanding maps is more general (and thus less rich) than the theory of smooth expanding maps given by Shub [Sh69] and Hirsch [Hirs70]. In particular, in this generality there is no classification of those X admitting an open distance-expanding map.

Chapter 4], [VO16, Chapter 11]); it is an open problem to determine which compact metric spaces admit open distance-expanding maps, though partial results abound (see e.g. [CM88, Gr81, Hira90a, Hira90b, Hirs70, Nek14, Nek20, Sh70]). Our consideration of the space $\mathcal{E}(X)$, equipped with the complete metric d_{Lip}, appears to be new, as is the exploitation of local connectedness (see Proposition 2.2 and Lemma 2.5).

We wish to establish the following result, which in particular implies Theorem A:

Theorem A′ (Joint TPO for expanding maps). *If X is a compact locally connected metric space and $\alpha \in (0,1]$, then the joint periodic locking set $\mathfrak{L}^\alpha(X)$ is an open and dense subset of $\mathcal{E}(X) \times C^{0,\alpha}(X)$.*

2.2. **Local connectedness and shadowing.** To prove Theorem A′, we require a number of preparatory results. Firstly, we show that local connectedness of X affords a degree of control over the surjectivity radius δ, a result which will ultimately lead to the key Locally Connected Shadowing Lemma (Lemma 2.5) on the existence of shadowing orbits for perturbed maps.

Proposition 2.2. *Let X be a compact locally connected metric space. Given $\gamma > 0$, there exists $\delta \in (0,\gamma)$ such that for all $\lambda > 1$, if $T \in \mathcal{E}(X)$ has expanding constants (γ,λ), then $B(T(x),\delta) \subseteq T(B(x,\delta/\lambda))$ for all $x \in X$.*

Proof. As X is compact and locally connected, there exists a finite open cover $\mathcal{U}$ of X consisting of connected open subsets, each with diameter smaller than γ. Let $\lambda > 1$, and suppose $T \in \mathcal{E}(X)$ has a pair of expanding constants (γ,λ). For each $y \in X$ and $U \in \mathcal{U}$ containing $x \coloneqq T(y)$, define $V \coloneqq T(B(y,\gamma)) \cap U$. As T is an open map, V is an open subset of X, therefore also an open subset of U. Since T is distance-expanding, if $z \in B(y,\gamma) \smallsetminus \overline{B}(y,\gamma/\lambda)$ then $d(x,T(z)) > \gamma$, so $T\bigl(B(y,\gamma) \smallsetminus \overline{B}(y,\gamma/\lambda)\bigr) \cap U = \emptyset$, therefore $V = T\bigl(\overline{B}(y,\gamma/\lambda)\bigr) \cap U$. Since T is continuous and $\overline{B}(y,\gamma/\lambda)$ is compact, $T\bigl(\overline{B}(y,\gamma/\lambda)\bigr)$ is also compact. As X is compact and Hausdorff, $T\bigl(\overline{B}(y,\gamma/\lambda)\bigr)$ is a closed subset of X, so V is also a closed subset of U. As U is connected and $V \neq \emptyset$, we have $V = U$.

Now let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$; clearly $\delta < \gamma$, since the diameters of all members of $\mathcal{U}$ are smaller than γ. For each $z \in X$, let $x \coloneqq T(z)$. For each $y \in B(x,\delta)$, there exists $U \in \mathcal{U}$ such that $x, y \in U$. As $U \subseteq T(B(z,\gamma))$, there exists $w \in T^{-1}(y) \cap B(z,\gamma)$, and the distance-expanding property of T means that $w \in B(z,\delta/\lambda)$. That is, $B(T(z),\delta) \subseteq T(B(z,\delta/\lambda))$ for any $z \in X$, as required. ☐

Lemma 2.3. *Let X be a compact metric space, and suppose $T_0 \in \mathcal{E}(X)$ has expanding constants (γ,λ). If $T \in \mathcal{E}(X)$ with $d_{\mathrm{Lip}}(T_0,T) < \lambda - 1$, then $(\gamma, \lambda - d_{\mathrm{Lip}}(T_0,T))$ is a pair of expanding constants for T.*

Proof. For all $x, y \in X$ with $d(x,y) < \gamma$, from the definition of d_{Lip} (cf. (2.1)) we see that

$$\begin{aligned} d(T(x),T(y)) &\geqslant d(T_0(x),T_0(y)) - |d(T_0(x),T_0(y)) - d(T(x),T(y))| \\ &> \lambda d(x,y) - d_{\mathrm{Lip}}(T_0,T)d(x,y) = (\lambda - d_{\mathrm{Lip}}(T_0,T))d(x,y), \end{aligned}$$

and $\lambda - d_{\mathrm{Lip}}(T_0,T) > \lambda - (\lambda - 1) = 1$, so indeed $(\gamma, \lambda - d_{\mathrm{Lip}}(T_0,T))$ is a pair of expanding constants for T, as required. ☐

Following [PU10], for any $\alpha > 0$ a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X will be called an *α-pseudo-orbit* if $d(x_{n+1},T(x_n)) \leqslant \alpha$ for all $n \in \mathbb{N}$. We say that $\{x_n\}_{n=0}^{k-1}$ is a *periodic α-pseudo-orbit* if $d(x_{n+1 \pmod k},T(x_n)) \leqslant \alpha$ for all $0 \leqslant n \leqslant k-1$. Given an α-pseudo-orbit $\{x_n\}_{n\in\mathbb{N}}$, and $\beta > 0$, a T-orbit $\{y_n\}_{n\in\mathbb{N}}$ satisfying $d(x_n,y_n) \leqslant \beta$ for all $n \in \mathbb{N}$ will be called a *β-shadowing orbit* of $\{x_n\}_{n\in\mathbb{N}}$; in this case we also say that $\{y_n\}_{n\in\mathbb{N}}$ *β-shadows* $\{x_n\}_{n\in\mathbb{N}}$.

We require the following result regarding the existence of shadowing orbits.[19]

[19]Note that the key hypothesis that $B(T(x),\delta) \subseteq T(B(x,\delta/\lambda))$ in our Lemma 2.4 is implicit rather than explicit in the statement of [PU10, Lemma 4.2.3], having been chosen earlier (in [PU10, Section 4.1]).

Lemma 2.4 ([PU10, Lemma 4.2.3])**.** *Let X be a compact metric space, and suppose $T \in \mathcal{E}(X)$ has a pair of expanding constants (δ, λ) satisfying $B(T(x), \delta) \subseteq T(B(x, \delta/\lambda))$ for all $x \in X$. If $\beta \in (0, \delta)$ and $\tau \in (0, \min\{(\lambda-1)\beta,\, \delta\}]$, then every τ-pseudo-orbit has a β-shadowing orbit.*

The following shadowing result, exploiting local connectedness of X, is fundamental to our method of proof in Theorem 2.7.

Lemma 2.5 (Locally Connected Shadowing Lemma). *Suppose X is a compact locally connected metric space, $T_0 \in \mathcal{E}(X)$ has expanding constants (γ, λ), and $\mathcal{O}_0$ is a periodic orbit for T_0. There exists $D = D(T_0, \mathcal{O}_0) > 0$ such that if $T \in \mathcal{E}(X)$ with $d_{\mathrm{Lip}}(T_0, T) < D$, then:*

(i) $\mathrm{LIP}(T) \leqslant 2\,\mathrm{LIP}(T_0)$.

(ii) $(\gamma, \lambda - d_{\mathrm{Lip}}(T_0, T))$ *is a pair of expanding constants for T.*

(iii) $\mathcal{O}_0$ *is a periodic $d_{\mathrm{Lip}}(T_0, T)$-pseudo-orbit for T.*

(iv) *There exists a T-periodic orbit $\mathcal{O}$ that $\left(\frac{d_{\mathrm{Lip}}(T_0,T)}{\lambda - d_{\mathrm{Lip}}(T_0,T)-1}\right)$-shadows $\mathcal{O}_0$, and whose minimum interpoint distance*[20] *satisfies $\Delta(\mathcal{O}) \geqslant \frac{1}{2}\Delta(\mathcal{O}_0)$.*

Proof. Let $\delta \in (0, \gamma)$ be as in Proposition 2.2, and define

$$D := \min\{\delta,\, \mathrm{LIP}(T_0),\, (\lambda-1)\delta/(1+\delta),\, (\lambda-1)/2,\, (\lambda-1)\Delta(\mathcal{O}_0)/(4+\Delta(\mathcal{O}_0))\}, \tag{2.3}$$

where we define the final term above to equal $+\infty$ when $\mathrm{card}\,\mathcal{O}_0 = 1$, i.e., when $\Delta(\mathcal{O}_0) = +\infty$. Suppose $T \in \mathcal{E}(X)$ is such that $d_{\mathrm{Lip}}(T_0, T) < D$.

(i) As $D \leqslant \mathrm{LIP}(T_0)$ (cf. (2.3)), by (2.1), for all $x, y \in X$ we have

$$\begin{aligned} d(T(x), T(y)) &\leqslant d(T_0(x), T_0(y)) + |d(T_0(x), T_0(y)) - d(T(x), T(y))| \\ &\leqslant (\mathrm{LIP}(T_0) + d_{\mathrm{Lip}}(T_0, T))d(x, y) \leqslant 2\,\mathrm{LIP}(T_0)d(x, y), \end{aligned}$$

and thus $\mathrm{LIP}(T) \leqslant 2\,\mathrm{LIP}(T_0)$, as required.

(ii) and (iii) Now $d_{\mathrm{Lip}}(T_0, T) < D \leqslant (\lambda-1)/2 < \lambda - 1$ (cf. (2.3)), so applying Lemma 2.3 gives that $(\gamma, \lambda - d_{\mathrm{Lip}}(T_0, T))$ is a pair of expanding constants for T, as required in (ii). Now $d(T_0(x), T(x)) \leqslant d_{\mathrm{Lip}}(T_0, T)$ for all $x \in X$, so $\mathcal{O}_0$ is a $d_{\mathrm{Lip}}(T_0, T)$-pseudo-orbit for T, as required in (iii).

(iv) Define $\lambda' := \lambda - d_{\mathrm{Lip}}(T_0, T)$, so that Proposition 2.2 gives

$$B(T(x), \delta) \subseteq T(B(x, \delta/\lambda')) \tag{2.4}$$

for all $x \in X$. Now $D \leqslant \delta$ and $D \leqslant \frac{(\lambda-1)\delta}{1+\delta}$ (cf. (2.3)), so $d_{\mathrm{Lip}}(T_0, T) < D \leqslant \delta$ and $d_{\mathrm{Lip}}(T_0, T) < D \leqslant (\lambda - D - 1)\delta < (\lambda' - 1)\delta$. Thus by (2.4), we can apply Lemma 2.4 to T and $\mathcal{O}_0$, so as to obtain the required T-periodic orbit $\mathcal{O}$ that $\frac{d_{\mathrm{Lip}}(T_0,T)}{\lambda'-1}$-shadows $\mathcal{O}_0$, with $\mathrm{card}\,\mathcal{O} \leqslant \mathrm{card}\,\mathcal{O}_0$. If $\mathrm{card}\,\mathcal{O} = 1$ then the minimum interpoint distance $\Delta(\mathcal{O})$ is by definition equal to $+\infty$ (cf. Subsection 1.5), so the inequality $\Delta(\mathcal{O}) \geqslant \frac{1}{2}\Delta(\mathcal{O}_0)$ is immediate. If $\mathrm{card}\,\mathcal{O} > 1$, then

$$\frac{d_{\mathrm{Lip}}(T_0, T)}{\lambda' - 1} < \frac{D}{\lambda - D - 1} \leqslant \frac{1}{4}\Delta(\mathcal{O}_0), \tag{2.5}$$

using the fact that $D \leqslant \frac{(\lambda-1)\Delta(\mathcal{O}_0)}{4+\Delta(\mathcal{O}_0)}$ (cf. (2.3)). Now $\mathcal{O}_0$ is $\frac{d_{\mathrm{Lip}}(T_0,T)}{\lambda'-1}$-shadowed by $\mathcal{O}$, so (2.5) implies that it is $\frac{1}{4}\Delta(\mathcal{O}_0)$-shadowed by $\mathcal{O}$. Hence, for distinct points $x, y \in \mathcal{O}$, there exist $x', y' \in \mathcal{O}_0$ with $d(x, x') \leqslant (1/4)\Delta(\mathcal{O}_0)$ and $d(y, y') \leqslant (1/4)\Delta(\mathcal{O}_0)$. Now $d(x', y') \geqslant \Delta(\mathcal{O}_0)$ by definition, so $d(x, y) \geqslant d(x', y') - d(x, x') - d(y, y') \geqslant (1/2)\Delta(\mathcal{O}_0)$, but x, y are arbitrary distinct members of $\mathcal{O}$, therefore $\Delta(\mathcal{O}) \geqslant (1/2)\Delta(\mathcal{O}_0)$, as required. □

[20]Recall (cf. Subsection 1.5) that $\Delta(F) := \min\{d(x, y) : x, y \in F, x \neq y\}$ if $\mathrm{card}\, F \geqslant 2$ and $\Delta(F) := +\infty$ if $\mathrm{card}\, F = 1$.

2.3. **A Mañé cohomology lemma.** Versions of the following Theorem 2.6, often referred to as the *Mañé Lemma*[21], are well known, but in the generality of our setting (i.e., open distance-expanding Lipschitz maps on arbitrary compact metric spaces), the semi-norm bound[22] (2.7) does not appear in the existing literature (see in particular [Bou11, Theorem 3.1]).

Theorem 2.6 (Mañé Lemma). *Let X be a compact metric space. Suppose $T \in \mathcal{E}(X)$ and $\alpha \in (0,1]$. There exists $L = L(T,\alpha) > 0$ such that for each $\phi \in C^{0,\alpha}(X)$, there exists $u \in C^{0,\alpha}(X)$ satisfying*

$$\psi \coloneqq \overline{\phi} + u - u \circ T \leqslant 0 \quad \text{and} \tag{2.6}$$

$$|u|_\alpha \leqslant L|\phi|_\alpha, \tag{2.7}$$

where $\overline{\phi} \coloneqq \phi - Q(T,\phi)$.

Proof. Let (γ,λ) be expanding constants for T. By [PU10, Lemma 4.1.2], there exists a constant $\delta > 0$ such that $B(T(x),\delta) \subseteq T(B(x,\gamma))$ for all $x \in X$. We may assume that $\delta < \gamma$. For $x \in X$, if $y \in X$ satisfies $\delta/\lambda \leqslant d(x,y) < \gamma$, then $d(T(x),T(y)) \geqslant \delta$, so $T(y) \notin B(T(x),\delta)$. This implies that $B(T(x),\delta) \subseteq T(B(x,\delta/\lambda))$ for all $x \in X$. Define the function u by

$$u(x) \coloneqq \sup_{n\in\mathbb{N}_0} \sup_{y \in T^{-n}(x)} S_n\overline{\phi}(y). \tag{2.8}$$

Note that $S_0\overline{\phi} \equiv 0$, which implies that $u \geqslant 0$. We will show that the function u has the required properties.

First we will show that u is well defined, i.e., always takes finite values. Let $\tau \coloneqq \min\{(\lambda-1)\delta/2,\ \delta\}$. Since X is compact, the maximal cardinality of a τ-separated subset of X is some $N \in \mathbb{N}$. Fixing $z \in X$ and $n \in \mathbb{N}_0$, we will recursively define two finite sequences of integers $i_1, i_2, \dots, i_m$ and $j_1, j_2, \dots, j_m$, both lying in the interval $[0, n-1]$.

Base step. Define $i_1 \coloneqq 0$.

Recursive step. Assume i_k is defined for some $k \in \mathbb{N}$. Let $j_k \in [0, n-1]$ be the largest integer satisfying $d\big(T^{i_k}(z), T^{j_k}(z)\big) < \tau$. Note that by definition we have $j_k \geqslant i_k$. If $j_k = n-1$, we complete the recursive step and set $m \coloneqq k$. If $j_k < n-1$, we define $i_{k+1} \coloneqq j_k + 1$. As the sequence $i_1, i_2, \dots$ is strictly increasing, this recursive step will terminate after finitely many steps.

The sequences $\{i_k\}_{k=1}^m$ and $\{j_k\}_{k=1}^m$ partition $\{0, 1, \dots, n-1\}$ into m sub-intervals $\{[i_k, j_k]\}_{k=1}^m$. We claim that $m \leqslant N$. Were this not the case then there would exist two distinct integers $k_1, k_2 \in [1,m]$, where without loss of generality $k_1 < k_2$, such that $d\big(T^{i_{k_1}}(x), T^{i_{k_2}}(x)\big) < \tau$, but $i_{k_2} \geqslant j_{k_1} + 1$, so this would contradict the definition of j_{k_1}.

Let k be an integer in $[1,m]$. We now estimate $S_{j_k - i_k}\overline{\phi}\big(T^{i_k}(z)\big)$, where without loss of generality $j_k > i_k$. Now $\delta < \gamma$, so (δ,λ) can also be viewed as a pair of expanding constants for T. For the periodic τ-pseudo orbit $\big\{T^{i_k}(z), T^{i_k+1}(z), \dots, T^{j_k-1}(z)\big\}$, applying Lemma 2.4 yields a periodic point w, of period $j_k - i_k$, such that if $0 \leqslant l \leqslant j_k - i_k - 1$ then $d\big(T^l(w), T^{i_k+l}(z)\big) \leqslant \delta/2 < \gamma$ and

$$d\big(T^l(w), T^{i_k+l}(z)\big) \leqslant \lambda^{-j_k+i_k+l+1} d\big(T^{j_k-i_k-1}(w), T^{j_k-1}(z)\big) \leqslant \lambda^{-j_k+i_k+l+1}\delta/2.$$

Consequently,

$$S_{j_k-i_k}\overline{\phi}\big(T^{i_k}(z)\big) \leqslant S_{j_k-i_k}\overline{\phi}(w) + |\phi|_\alpha \sum_{l=0}^{j_k-i_k-1} (\delta/2)^\alpha \lambda^{\alpha(-j_k+i_k+l+1)} \leqslant \frac{\delta^\alpha \lambda^\alpha |\phi|_\alpha}{2^\alpha(\lambda^\alpha - 1)}. \tag{2.9}$$

[21]The term *Mañé lemma* originated in [Bou00], in view of the resemblance to a result of Mañé [Man96, Theorem B] in the context of Lagrangian flows.

[22]The bound (2.7) on the semi-norm of the sub-action u, by a fixed constant multiple of the semi-norm of ϕ, will be important in our subsequent proof of Joint TPO, where both the map T and function ϕ vary; in the proofs of Individual TPO, as in [Boc19, Co16, HLMXZ19], this additional control is not required.

Since $m \leqslant N$, (2.9) implies that

$$S_n\overline{\phi}(z) = \sum_{k=1}^{m}\Big(S_{j_k-i_k}\overline{\phi}\big(T^{i_k}(z)\big) + \overline{\phi}\big(T^{j_k}(z)\big)\Big) \leqslant N\bigg(\frac{\delta^\alpha\lambda^\alpha|\phi|_\alpha}{2^\alpha(\lambda^\alpha-1)} + \|\overline{\phi}\|_\infty\bigg). \tag{2.10}$$

Since $z \in X$ and $n \in \mathbb{N}_0$ were arbitrary, (2.10) implies that u is everywhere finite, as required.

Next, we will show that u satisfies (2.6). By the definition of u (cf. (2.8)), for each $x \in X$,

$$\begin{aligned} u(x) + \overline{\phi}(x) &= \sup_{n\geqslant 0}\sup_{y\in T^{-n}(x)} S_{n+1}\overline{\phi}(y) \leqslant \sup_{n\geqslant 0}\sup_{y\in T^{-n-1}(T(x))} S_{n+1}\overline{\phi}(y) \\ &= \sup_{n\geqslant 1}\sup_{y\in T^{-n}(T(x))} S_n\overline{\phi}(y) \leqslant u(T(x)), \end{aligned}$$

which is the desired (2.6).

Lastly, we will establish the bound (2.7). Fixing x, $y \in X$, we consider two cases.

Case 1. Suppose $d(x,y) < \delta$. If $z_1 \in T^{-1}(x)$, the fact that $B(T(z_1),\delta) \subseteq T(B(z_1,\gamma))$ means there exists $w_1 \in T^{-1}(y) \cap B(z_1,\gamma)$. As (γ,λ) is a pair of expanding constants for T, then $d(z_1,w_1) < \lambda^{-1}d(x,y)$. Iterating this argument, if $n \in \mathbb{N}$ and $z_n \in T^{-n}(x)$ then there exists $w_n \in T^{-n}(y)$ with $d(T^m(z_n),T^m(w_n)) < \lambda^{m-n}d(x,y)$ for all $0 \leqslant m \leqslant n-1$, so

$$S_n\overline{\phi}(z_n) \leqslant S_n\overline{\phi}(w_n) + |\phi|_\alpha\sum_{i=1}^{n}\lambda^{-i\alpha}d(x,y)^\alpha \leqslant u(y) + |\phi|_\alpha\frac{d(x,y)^\alpha}{\lambda^\alpha-1}.$$

Note that if $n=0$ then $S_0\overline{\phi}(x) = 0 \leqslant u(y) \leqslant u(y) + |\phi|_\alpha\frac{1}{\lambda^\alpha-1}d(x,y)^\alpha$. Since $n \in \mathbb{N}_0$ and $z_n \in T^{-n}(x)$ were chosen arbitrarily, we see that

$$u(x) \leqslant u(y) + |\phi|_\alpha d(x,y)^\alpha/(\lambda^\alpha-1). \tag{2.11}$$

Since the analogue of (2.11), with x and y interchanged, can be proved in the same way, it follows that

$$|u(x)-u(y)| \leqslant |\phi|_\alpha d(x,y)^\alpha/(\lambda^\alpha-1). \tag{2.12}$$

Case 2. Suppose $d(x,y) \geqslant \delta$. Now $u \geqslant 0$, so

$$|u(x)-u(y)| \leqslant \sup_{z\in X}u(z) \leqslant \delta^{-\alpha}\cdot\sup_{z\in X}u(z)\cdot d(x,y)^\alpha. \tag{2.13}$$

We now estimate $\sup_{z\in X}u(z)$. Since $Q\big(T,\overline{\phi}\big) = 0$ and $\overline{\phi}$ is continuous, it follows that $\max_{y\in X}\overline{\phi}(y) \geqslant 0$ and $\min_{y\in X}\overline{\phi}(y) \leqslant 0$. But $\max_{y\in X}\overline{\phi}(y) - \min_{y\in X}\overline{\phi}(y) \leqslant |\phi|_\alpha(\operatorname{diam} X)^\alpha$, therefore

$$\big\|\overline{\phi}\big\|_\infty \leqslant |\phi|_\alpha(\operatorname{diam} X)^\alpha. \tag{2.14}$$

Combining (2.10) and (2.14), together with the definition of u (cf. (2.8)), we see that

$$\sup_{z\in X}u(z) \leqslant N|\phi|_\alpha\bigg(\frac{\delta^\alpha\lambda^\alpha}{2^\alpha(\lambda^\alpha-1)} + (\operatorname{diam} X)^\alpha\bigg). \tag{2.15}$$

Combining (2.15) and (2.13), and then (2.12), we see that the required bound (2.7) holds for the choice

$$L := \max\bigg\{\frac{1}{\lambda^\alpha-1},\ \delta^{-\alpha}N\bigg(\frac{\delta^\alpha\lambda^\alpha}{2^\alpha(\lambda^\alpha-1)} + (\operatorname{diam} X)^\alpha\bigg)\bigg\}. \qquad \square$$

2.4. **Joint perturbation.** The following Theorem 2.7 is the key result that will allow us to prove Theorem A′ (and hence Theorem A). The underlying *joint perturbation* idea will be a motif throughout the article, re-occurring in different forms in the context of Anosov diffeomorphisms (see Theorems 2.12 and 2.13), and beta-transformations (see Theorem 6.4).

Theorem 2.7 (Joint Perturbation: expanding maps). *Let X be a compact locally connected metric space. Suppose $T_0 \in \mathcal{E}(X)$ has expanding constants (γ, λ), and $\alpha \in (0,1]$. Let $\mathcal{O}_0$ be a T_0-periodic orbit. Let $D = D(T_0, \mathcal{O}_0) > 0$ be as in Lemma 2.5, and assume that $D < \min\{1, \lambda - 1\}$. Then there exists $C > 0$ such that for all $T \in \mathcal{E}(X)$ with $d_{\mathrm{Lip}}(T_0, T) < D$, there exists a T-periodic orbit $\mathcal{O}$ such that for all $\phi \in C^{0,\alpha}(X)$ with $\mathcal{M}_{\max}(T_0, \phi) = \{\mu_{\mathcal{O}_0}\}$, the unique T-maximizing measure for the function $\phi - 2C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha$ is $\mu_{\mathcal{O}}$.*

Proof. Let $\delta \in (0, \gamma)$ be as in Proposition 2.2. Let $L = L(T_0, \alpha) > 0$ be as in the Mañé Lemma (Theorem 2.6). Define constants

$$p \coloneqq \operatorname{card} \mathcal{O}_0, \tag{2.16}$$

$$r \coloneqq \min\{\Delta(\mathcal{O}_0)/4,\, \gamma\}, \tag{2.17}$$

$$L_1 \coloneqq 1 + ((\lambda - D)^\alpha - 1)^{-1} + 2L\,(2\,\mathrm{LIP}(T_0))^\alpha, \tag{2.18}$$

$$L_2 \coloneqq L + (\lambda - D - 1)^{-\alpha}, \tag{2.19}$$

$$C \coloneqq \max\{1,\, L_2(1 + p + L_1)(2\,\mathrm{LIP}(T_0)/r)^\alpha\}. \tag{2.20}$$

Fix $\phi \in C^{0,\alpha}(X)$ with $\mathcal{M}_{\max}(T_0, \phi) = \{\mu_{\mathcal{O}_0}\}$. By the Mañé Lemma (Theorem 2.6) applied to the map T_0, there exist $u, \psi \in C^{0,\alpha}(X)$ satisfying (2.6) and (2.7), in other words,

$$\psi \coloneqq \overline{\phi} + u - u \circ T_0 \leqslant 0, \tag{2.21}$$

where we recall that $\overline{\phi}$ denotes $\phi - Q(T_0, \phi)$, and

$$|u|_\alpha \leqslant L|\phi|_\alpha. \tag{2.22}$$

Now suppose $T \in \mathcal{E}(X)$ is such that $d_{\mathrm{Lip}}(T_0, T) < D$, and let $\mathcal{O}$ be the periodic orbit from Lemma 2.5 (iv). Combining (2.17) and Lemma 2.5 (iv), we see that

$$r \leqslant \Delta(\mathcal{O})/2. \tag{2.23}$$

Now define $\psi' \coloneqq \overline{\phi} + u - u \circ T$ and

$$\tau \coloneqq L|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^\alpha, \tag{2.24}$$

so that (2.21) and (2.22) give

$$\psi'(x) \leqslant \psi(x) + |u(T_0(x)) - u(T(x))| \leqslant L|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^\alpha = \tau \quad \text{for all } x \in X. \tag{2.25}$$

Define functions

$$\xi \coloneqq \overline{\phi} - C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha \quad \text{and} \tag{2.26}$$

$$\xi' \coloneqq \xi + u - u \circ T = \psi' - C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha, \tag{2.27}$$

where the second equality in (2.27) follows from (2.26) and the definition of ψ'. Define $\eta \coloneqq \int \xi' \,\mathrm{d}\mu_{\mathcal{O}}$, so that (2.27), (2.26), Lemma 2.5 (iv), and the fact that $\int \overline{\phi} \,\mathrm{d}\mu_{\mathcal{O}_0} = Q(T_0, \overline{\phi}) = 0$, give

$$\begin{aligned} \eta &= \int \xi' \,\mathrm{d}\mu_{\mathcal{O}} = \int \xi \,\mathrm{d}\mu_{\mathcal{O}} = \int \overline{\phi} \,\mathrm{d}\mu_{\mathcal{O}} \\ &\geqslant \int \overline{\phi} \,\mathrm{d}\mu_{\mathcal{O}_0} - |\phi|_\alpha \biggl(\frac{d_{\mathrm{Lip}}(T_0, T)}{\lambda - d_{\mathrm{Lip}}(T_0, T) - 1} \biggr)^\alpha \geqslant -|\phi|_\alpha \biggl(\frac{d_{\mathrm{Lip}}(T_0, T)}{\lambda - D - 1} \biggr)^\alpha. \end{aligned} \tag{2.28}$$

Combining (2.24), (2.28), and recalling that $L_2 = L + (\lambda - D - 1)^{-\alpha}$ (cf. (2.19)), we obtain

$$\tau - \eta \leqslant L_2|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^\alpha. \tag{2.29}$$

We wish to prove that $\mu_{\mathcal{O}} \in \mathcal{M}_{\max}(T, \xi)$, which is equivalent to showing that $\mu_{\mathcal{O}} \in \mathcal{M}_{\max}(T, \xi')$, and by [Je19, Proposition 2.2] it suffices to establish that

$$\liminf_{n \to +\infty} \frac{1}{n} S_n^T \xi'(x) \leqslant \eta \quad \text{for all } x \in X. \tag{2.30}$$

Define

$$\rho := \big(C|\phi|_\alpha d_{\mathrm{Lip}}(T_0,T)^{\alpha/2}/(\tau-\eta)\big)^{-1/\alpha}. \tag{2.31}$$

By (2.25) and (2.27),

$$\xi'(x) \leqslant \eta \quad \text{if } x \notin B(\mathcal{O},\rho). \tag{2.32}$$

Fixing $x \in X$, we will recursively construct two sequences $\{x_t\}_{t\in\mathbb{N}}$ in X and $\{n_t\}_{t\in\mathbb{N}}$ in $\mathbb{N}$, with the properties that $x_{t+1} = T^{n_t}(x_t)$ and

$$S^T_{n_t}\xi'(x_t) \leqslant n_t\eta \quad \text{for all } t \in \mathbb{N}. \tag{2.33}$$

Base step. Define $x_1 := x$.

Recursive step. Assume that for some $t \in \mathbb{N}$, $\{x_i\}_{i=1}^t$ and $\{n_i\}_{i=1}^{t-1}$ are defined. Consider the following three cases.

Case A. Assume $x_t \in \mathcal{O}$. Define $n_t := p$ and $x_{t+1} := T^p(x_t)$. Then (2.33) follows from the definition of η.

Case B. Assume $x \notin B(\mathcal{O},\rho)$. Define $n_t := 1$ and $x_{t+1} := T(x_t)$, so that (2.33) follows from (2.32).

Case C. Assume $x \in B(\mathcal{O},\rho) \smallsetminus \mathcal{O}$. Using (2.31), (2.29), the fact that $C \geqslant L_2 D^{\alpha/2}(2\,\mathrm{LIP}(T_0)/r)^\alpha$ (cf. (2.20)), the fact that $d_{\mathrm{Lip}}(T_0,T) < D$, and Lemma 2.5 (i), we obtain

$$\rho \leqslant r/(2\,\mathrm{LIP}(T_0)) \leqslant r/\mathrm{LIP}(T). \tag{2.34}$$

Let $y \in \mathcal{O}$ be such that $d(x_t,\mathcal{O}) = d(x_t,y)$, and define

$$N := \min\big\{i \in \mathbb{N}_0 : d\big(T^{i+1}(x_t), T^{i+1}(y)\big) \geqslant r\big\}, \tag{2.35}$$
$$m := \min\big\{i \in \mathbb{N}_0 : d\big(T^{i}(x_t), T^{i}(y)\big) \geqslant \rho\big\}. \tag{2.36}$$

The existence of N follows from the fact that $r \leqslant \gamma$ (cf. (2.17)) and the existence of m follows from $\rho < r$ (by (2.34) and $\mathrm{LIP}(T) > 1$). By (2.17), (2.23), and Lemma 2.5 (ii), for $0 \leqslant i \leqslant N$,

$$d\big(T^i(x_t),\mathcal{O}\big) = d\big(T^i(x_t),T^i(y)\big) \leqslant \frac{d(T^{i+1}(x_t),T^{i+1}(y))}{\lambda - d_{\mathrm{Lip}}(T_0,T)} < \frac{d(T^{i+1}(x_t),T^{i+1}(y))}{\lambda - D}. \tag{2.37}$$

By (2.34), m exists and satisfies

$$1 \leqslant m \leqslant N. \tag{2.38}$$

In this case, we define $n_t := N+1$ and $x_{t+1} := T^{N+1}(x_t)$.

Next, we estimate $S^T_{n_t}\xi'(x_t)$ so as to deduce (2.33). Noting that $\xi' \leqslant \psi'$, direct calculation gives

$$S^T_{n_t}\xi'(x_t) \leqslant S^T_m\psi'(y) + \big|S^T_m\psi'(y) - S^T_m\psi'(x_t)\big| + S^T_{N-m}\xi'(T^m(x_t)) + \xi'\big(T^N(x_t)\big). \tag{2.39}$$

We now estimate the four terms on the right-hand side of (2.39). For the first term, write $m = qp + l$ for some $q \in \mathbb{N}_0$ and $0 \leqslant l \leqslant p-1$, recalling that $p = \operatorname{card}\mathcal{O}_0$ (cf. (2.16)). Using (2.25),

$$S^T_m\psi'(y) = qS^T_p\psi'(y) + S^T_l\psi'(y) \leqslant qp\eta + l\tau = m\eta + l(\tau-\eta) \leqslant m\eta + p(\tau-\eta). \tag{2.40}$$

For the second term on the right-hand side of (2.39), note that (2.38), Lemma 2.5 (i), and (2.36) give

$$d(x_t,y) < d(T^m(x_t),T^m(y)) \leqslant \mathrm{LIP}(T)d\big(T^{m-1}(x_t),T^{m-1}(y)\big) < 2\,\mathrm{LIP}(T_0)\rho.$$

Thus, by (2.26), (2.37), (2.7), and (2.18), we estimate

$$\begin{aligned}\left|S_m^T\psi'(y) - S_m^T\psi'(x_t)\right| &\leqslant \left|S_m^T\overline{\phi}(y) - S_m^T\overline{\phi}(x_t)\right| + |u(x_t) - u(y)| + |u(T^m(x_t)) - u(T^m(y))| \\ &\leqslant |\phi|_\alpha \sum_{i=0}^{m-1} d(T^i(x_t), T^i(y))^\alpha + 2|u|_\alpha (2\rho \operatorname{LIP}(T_0))^\alpha \\ &\leqslant |\phi|_\alpha \rho^\alpha \sum_{i=0}^{m-1} (\lambda - D)^{-i\alpha} + 2L|\phi|_\alpha (2\rho \operatorname{LIP}(T_0))^\alpha \\ &\leqslant |\phi|_\alpha \rho^\alpha \bigl(1 + ((\lambda - D)^\alpha - 1)^{-1} + 2L(2\operatorname{LIP}(T_0))^\alpha\bigr) \\ &= L_1 |\phi|_\alpha \rho^\alpha.\end{aligned} \tag{2.41}$$

For the third term on the right-hand side of (2.39), by (2.36), (2.37), and (2.32) we have

$$S_{N-m}^T \xi'(T^m(x_t)) \leqslant (N - m)\eta, \tag{2.42}$$

while for the fourth term, by (2.27), (2.25), (2.35), and Lemma 2.5 (i), we have

$$\xi'\bigl(T^N(x_t)\bigr) \leqslant \tau - C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} (2\operatorname{LIP}(T_0)/r)^{-\alpha}. \tag{2.43}$$

Finally, note that $d_{\mathrm{Lip}}(T_0, T) < D < 1$, as this was a hypothesis. Combining (2.39), (2.40), (2.41), (2.42), and (2.43), using $C = \max\{1,\, L_2(1 + p + L_1)(2\operatorname{LIP}(T_0)/r)^\alpha\}$ (cf. (2.20)), (2.31), and (2.29), and writing $L_3 \coloneqq d_{\mathrm{Lip}}(T_0, T)$ for simplicity, we have

$$\begin{aligned} S_{n_t}^T \xi'(x_t) - n_t \eta &\leqslant (p+1)(\tau - \eta) + L_1 |\phi|_\alpha \rho^\alpha - C|\phi|_\alpha L_3^{\alpha/2} (2\operatorname{LIP}(T_0)/r)^{-\alpha} \\ &\leqslant (\tau - \eta)\bigl(1 + p + L_1 L_3^{-\alpha/2}\bigr) - C|\phi|_\alpha L_3^{\alpha/2} (2\operatorname{LIP}(T_0)/r)^{-\alpha} \\ &\leqslant (\tau - \eta) L_3^{-\alpha/2} (1 + p + L_1) - C|\phi|_\alpha L_3^{\alpha/2} (2\operatorname{LIP}(T_0)/r)^{-\alpha} \\ &\leqslant L_2 |\phi|_\alpha L_3^{\alpha/2} (1 + p + L_1) - C|\phi|_\alpha L_3^{\alpha/2} (2\operatorname{LIP}(T_0)/r)^{-\alpha} \\ &\leqslant 0. \end{aligned}$$

So the required inequality (2.33) holds, and therefore the recursive step is complete.

For each $t \in \mathbb{N}$, set $N_t \coloneqq \sum_{i=1}^t n_i$. By (2.33),

$$\liminf_{n\to+\infty} \frac{1}{n} S_n^T \xi'(x) \leqslant \liminf_{t\to+\infty} \frac{1}{N_t} S_{N_t}^T \xi'(x) = \liminf_{t\to+\infty} \frac{1}{N_t} \sum_{i=1}^t S_{n_i}^T \xi'(x_t) \leqslant \frac{\eta}{N_t} \sum_{i=1}^t n_i = \eta.$$

Thus, (2.30) holds, so $\mu_{\mathcal{O}} \in \mathcal{M}_{\max}(T, \xi)$. Then by (2.26), $\mu_{\mathcal{O}}$ is the unique T-maximizing measure for the function

$$\phi - 2C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha = \xi + Q(T, \phi) - C|\phi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha,$$

and so the proof of the theorem is complete. □

We are now almost ready to prove Theorem A′, where in particular the Joint Perturbation Theorem (Theorem 2.7) will be used to show that $\mathfrak{L}^\alpha(X)$ is *open* in $\mathcal{E}(X) \times C^{0,\alpha}(X)$. To show that $\mathfrak{L}^\alpha(X)$ is *dense*, however, we require the following individual typical periodic optimization theorem due to Contreras [Co16, Theorem A] (the statement below is closer to the reformulation of Bochi [Boc19]):

Theorem 2.8 (Individual TPO for expanding maps [Co16]**).** *If X is a compact metric space and $T \in \mathcal{E}(X)$, and $\alpha \in (0, 1]$, then* $\operatorname{Lock}^\alpha(T)$ *is an open and dense subset of $C^{0,\alpha}(X)$.*

Remark 2.9. The definition of *expanding* used in [Co16] corresponds precisely to our definition of *open distance-expanding and Lipschitz continuous.*

Proof of Theorem A′. Let $T_0 \in \mathcal{E}(X)$, $\alpha \in (0,1]$, and $\phi \in \mathrm{Lock}^\alpha(T_0)$. In this proof, we write $B(\phi, r) := \{\psi \in C^{0,\alpha}(X) : \|\phi - \psi\|_\alpha < r\}$ and $B_{\mathrm{Lip}}(T_0, r) := \{T \in \mathcal{E}(X) : d_{\mathrm{Lip}}(T_0, T) < r\}$ for $r \geqslant 0$.

Let $\mathcal{O}_0$ be the T_0-periodic orbit such that $\mathcal{M}_{\max}(T_0, \phi) = \{\mu_{\mathcal{O}_0}\}$. Then there exists $\theta > 0$ such that if $\psi \in C^{0,\alpha}(X)$ with $\|\psi - \phi\|_\alpha < \theta$ then $\mathcal{M}_{\max}(T_0, \psi) = \{\mu_{\mathcal{O}_0}\}$. Now $|\cdot|_\alpha \leqslant \|\cdot\|_\alpha$, and $|\cdot|_\alpha$ is sub-additive, so we can assume without loss of generality that θ is small enough (i.e., $\theta \leqslant 2^{-1}|\phi|_\alpha$) such that

$$\frac{1}{2}|\phi|_\alpha \leqslant |\psi|_\alpha \leqslant \frac{3}{2}|\phi|_\alpha \quad \text{for all } \psi \in B(\phi, \theta). \tag{2.44}$$

Applying Theorem 2.7 (to T_0 and $\mathcal{O}_0$), let C, $D > 0$ be as in that theorem. Define

$$E := \min\bigl\{D,\ (18C(1 + \mathrm{diam}(X)^\alpha)|\phi|_\alpha/\theta)^{-2/\alpha}\bigr\}. \tag{2.45}$$

Now fix $T \in B_{\mathrm{Lip}}(T_0, E)$ and $\psi \in B(\phi, \theta/2)$. Now $d_{\mathrm{Lip}}(T_0, T) < E \leqslant D$, so let $\mathcal{O}$ be the T-periodic orbit whose existence is guaranteed by Theorem 2.7. Set

$$\psi' := \psi + 6C|\psi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha. \tag{2.46}$$

Note that $\|d(\cdot, \mathcal{O})\|_\alpha = \|d(\cdot, \mathcal{O})\|_\infty + |d(\cdot, \mathcal{O})|_\alpha \leqslant 1 + \mathrm{diam}(X)^\alpha$.

Thus, by (2.46), (2.44), and (2.45), we obtain

$$\|\psi - \psi'\|_\alpha \leqslant 6C(1 + \mathrm{diam}(X)^\alpha)|\psi|_\alpha E^{\alpha/2} \leqslant 9C(1 + \mathrm{diam}(X)^\alpha)|\phi|_\alpha E^{\alpha/2} \leqslant \theta/2,$$

and together with the fact that $\psi \in B(\phi, \theta/2)$, we deduce that $\psi' \in B(\phi, \theta)$. So both ψ and ψ' belong to $B(\phi, \theta)$, thus by (2.44) we deduce that

$$6C|\psi|_\alpha \geqslant 3C|\phi|_\alpha \geqslant 2C|\psi'|_\alpha. \tag{2.47}$$

Now Theorem 2.7 asserts that $\mu_{\mathcal{O}}$ is the unique T-maximizing measure for the function

$$\psi' - 2C|\psi'|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha,$$

so by (2.47), together with the fact that $\mu_{\mathcal{O}}$ is the unique T-maximizing measure for $-d(\cdot, \mathcal{O})^\alpha$, we see that $\mu_{\mathcal{O}}$ is also the unique T-maximizing measure for the function

$$\psi = \psi' - 6C|\psi|_\alpha d_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha.$$

Therefore we have established that the open neighbourhood $B_{\mathrm{Lip}}(T_0, E) \times B(\phi, \theta/2)$ of (T_0, ϕ) is contained in $\mathfrak{L}^\alpha(X)$. So $\mathfrak{L}^\alpha(X)$ is an open subset of $\mathcal{E}(X) \times C^{0,\alpha}(X)$.

The fact that $\mathfrak{L}^\alpha(X)$ is dense in $\mathcal{E}(X) \times C^{0,\alpha}(X)$ follows readily (by a straightforward topological argument) from the fact (see Theorem 2.8) that for each $T \in \mathcal{E}(X)$, the fibre $\{T\} \times \mathrm{Lock}^\alpha(T)$ is dense in $\{T\} \times C^{0,\alpha}(X)$.

So $\mathfrak{L}^\alpha(X)$ is both open and dense in $\mathcal{E}(X) \times C^{0,\alpha}(X)$, therefore the theorem is proved. □

As mentioned in Section 1, a class of expanding maps of particular interest consists of those on compact Riemannian manifolds (see e.g. the exposition in [URM22, Chapter 6]), as investigated initially by Shub [Sh69, Sh70], and later notably by Gromov [Gr81]. Our Joint TPO result for expanding maps has an immediate corollary in the setting of compact Riemannian manifolds:

Theorem 2.10 (Joint TPO for expanding maps on manifolds). *Suppose M is a compact Riemannian manifold, with distance function induced by the Riemannian metric, and $\alpha \in (0,1]$. There is an open dense subset of pairs $(T, \phi) \in \mathcal{E}(M) \times C^{0,\alpha}(M)$ with the periodic optimization property.*

Remark 2.11. As mentioned in Subsection 1.2, the fact that Theorem A implies *joint typical uniqueness* of the maximizing measure has an interpretation in terms of a large deviation principle for zero-temperature limits of equilibrium states. Specifically, if $\mu_{T,t\phi}$ denotes an equilibrium state for the map T and potential $t\phi$, then for an open dense subset of pairs $(T, \phi) \in \mathcal{E}(X) \times C^{0,\alpha}(X)$, the family $\{\mu_{T,t\phi}\}_{t \in (1,+\infty)}$ satisfies the large deviation principle as $t \to +\infty$, in other words there exists a lower semi-continuous function $I \colon X \to [0, +\infty]$ such that $\liminf_{t\to+\infty} \frac{1}{t} \log \mu_{T,t\phi}(\mathcal{G}) \geqslant -\inf_{x\in\mathcal{G}} I(x)$

if $\mathcal{G} \subseteq X$ is open, and $\limsup_{t\to+\infty} \frac{1}{t} \log \mu_{T,t\phi}(\mathcal{K}) \leqslant -\inf_{x\in\mathcal{K}} I(x)$ if $\mathcal{K} \subseteq X$ is closed. This follows from results in [LiSu24] (cf. [Ana04, Wan19] for related phenomena in the zero-temperature limit of, respectively, Lagrangian dynamics and SLE).

2.5. **Anosov diffeomorphisms.** Here we will prove joint typical periodic optimization for the space of C^1 Anosov diffeomorphisms.[23] Recall that for a smooth compact Riemannian manifold M, if a diffeomorphism $f : M \to M$ has a hyperbolic structure on all of M then it is said to be Anosov (see e.g. [KH95, Ni71, Robi95, Wen16]). We require the following joint perturbation result:

Theorem 2.12 (Joint Perturbation: Anosov diffeomorphisms). *Let M be a smooth compact Riemannian manifold, with distance function d induced by the Riemannian metric, and let $\mathcal{A}(M)$ be the space of C^1 Anosov diffeomorphisms on M, equipped with the C^1 topology. Let $\alpha \in (0,1]$. Let $f \in \mathcal{A}(M)$, and suppose $\mathcal{O}$ is an f-periodic orbit. There exists a neighbourhood $U \subseteq \mathcal{A}(M)$ of f, and $C > 0$, such that for all $g \in U$, and all $\phi \in C^{0,\alpha}(M)$ with $\mathcal{M}_{\max}(f, \phi) = \{\mu_{\mathcal{O}}\}$, there exists a topological conjugacy h_g with $h_g \circ f = g \circ h_g$, and if $\mathcal{O}_g := h_g(\mathcal{O})$ then $\mu_{\mathcal{O}_g}$ is the unique g-maximizing measure for the function $\phi - 2C|\phi|_\alpha d_\infty(h_g, \mathrm{id})^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha$, where $d_\infty(h_g, \mathrm{id}) := \max_{x\in M} d(h_g(x), x)$.*

Proof. The proof of this result includes aspects that are similar in spirit to those for the joint perturbation theorem for expanding maps (Theorem 2.7) and beta-transformations (Theorem 6.4), and we reflect this by wherever possible adopting analogous notation. Since the proofs of Theorems 2.7 and 6.4 are given in full detail, and in order to minimise repetition, we therefore only indicate the general strategy of the proof here, omitting calculations which closely resemble those given in the proofs of Theorems 2.7 and 6.4.

Using e.g. [Wen16, Theorem 4.6, Lemmas 4.8, and 4.11] it can be shown that for all $f \in \mathcal{A}(M)$, there exist $K, \delta > 0$, $\lambda > 1$, and a neighbourhood U of f in $\mathcal{A}(M)$, such that if $g \in U$, $n \in \mathbb{N}$, and $\max_{0\leqslant i\leqslant n} d\big(g^i(x), g^i(y)\big) < \delta$, then

$$d\big(g^i(x), g^i(y)\big) \leqslant K\lambda^{-\min\{i,\, n-i\}}(d(x, y) + d(g^n(x), g^n(y)) \quad \text{for all } 0 \leqslant i \leqslant n. \tag{2.48}$$

Moreover $\mathcal{A}(M)$ has structural stability (see e.g. [Wen16, Theorems 4.19 and 4.20]): the above U can be chosen such that if $g \in U$ then $\mathrm{LIP}(g) \leqslant 2\,\mathrm{LIP}(f)$, and there is a homeomorphism $h_g : M \to M$ with $h_g \circ f = g \circ h_g$, and $d_\infty(h_g, \mathrm{id}) < \min\{\Delta(\mathcal{O})/4, 1\}$, which implies $\Delta(\mathcal{O}_g) \geqslant \Delta(\mathcal{O})/2$. For Anosov diffeomorphisms there is also a Mañé lemma with semi-norm control (cf. [Bou11, Theorem 3.1 and Section 4]): there exists $L > 0$ (depending on f) such that given $\phi \in C^{0,\alpha}(M)$, there exists $u \in C^{0,\alpha}(M)$ with $|u|_\alpha \leqslant L|\phi|_\alpha$ and $\psi := \overline{\phi} + u - u \circ f \leqslant 0$. Defining $r := \min\{\Delta(\mathcal{O})/(8\,\mathrm{LIP}(f)), \delta\}$, we have $r \leqslant \Delta(\mathcal{O})/(8\,\mathrm{LIP}(f)) \leqslant \Delta(\mathcal{O}_g)/(4\,\mathrm{LIP}(f)) \leqslant \Delta(\mathcal{O}_g)/(2\,\mathrm{LIP}(g))$.

Defining $\psi_g := \overline{\phi} + u - u \circ g$, we estimate $\psi_g \leqslant \tau := \mathrm{O}(|\phi|_\alpha d_\infty(h_g, \mathrm{id})^\alpha)$ (cf. (2.25)); note, however, that unlike in (2.25), here we estimate ψ_g by using that

$$d_\infty(f, g) \leqslant d_\infty(f, h_g \circ f) + d_\infty(g, g \circ h_g) \leqslant (1 + \mathrm{LIP}(g))d_\infty(h_g, \mathrm{id}) \leqslant (1 + 2\,\mathrm{LIP}(f))d_\infty(h_g, \mathrm{id}),$$

and therefore $\|u \circ f - u \circ g\|_\infty \leqslant L|\phi|_\alpha(1 + 2\,\mathrm{LIP}(f))^\alpha d_\infty(h_g, \mathrm{id})^\alpha$. Defining $\eta := \int \psi \,\mathrm{d}\mu_{\mathcal{O}_g}$ we see that $\eta = -\mathrm{O}(|\phi|_\alpha d_\infty(h_g, \mathrm{id})^\alpha)$ (cf. (2.28)). For $C > 0$, define $\phi_C := \phi - C|\phi|_\alpha d_\infty(h_g, \mathrm{id})^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha$ and $\psi_C := \psi_g - C|\phi|_\alpha d_\infty(h_g, \mathrm{id})^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha$, and note that $\eta = \int \psi_C \,\mathrm{d}\mu_{\mathcal{O}_g}$.

We wish to show that $\mu_{\mathcal{O}_g}$ is (g, ψ_C)-maximizing, i.e., (cf. [Je19, Proposition 2.2]) that for $x \in M$,

$$\liminf_{n\to+\infty} \frac{1}{n} S^g_n \psi_C(x) \leqslant \eta. \tag{2.49}$$

Define $\rho := \Big(\frac{\tau-\eta}{C|\phi|_\alpha d_\infty(h_g,\mathrm{id})^{\alpha/2}}\Big)^{1/\alpha}$, to make $\psi_C(x) \leqslant \eta$ if $x \notin B(\mathcal{O}_g, \rho)$. For $x \in M$ we recursively define sequences $\{x_t\}_{t\in\mathbb{N}}$ and $\{n_t\}_{t\in\mathbb{N}}$, such that $x_{t+1} = g^{n_t}(x_t)$ for all t. Set $x_1 := x$, then as a recursive step assume, for $t \in \mathbb{N}$, that $\{x_i\}_{i=1}^{t}$ and $\{n_i\}_{i=1}^{t-1}$ are defined, and consider the following three (exhaustive

[23] Note that the treatment in this subsection is more streamlined and less detailed than the proofs of Joint TPO for expanding maps and beta-transformations, on the one hand so as to minimise repetition of similar arguments, on the other hand in view of a forthcoming more comprehensive study of Joint TPO for general hyperbolic systems.

and mutually exclusive) cases. As a first case, if $x_t \notin B(\mathcal{O}_g, \rho)$ then let $n_t := 1$ and $x_{t+1} := g(x_t)$. As a second case (in the purely expanding context of Theorem 2.7, this corresponds to the trivial case $x_t \in \mathcal{O}_g$), suppose that $\mathcal{O}^g(x_t) \subseteq B(\mathcal{O}_g, r)$. Let $y \in \mathcal{O}_g$ be such that $d(x_t, \mathcal{O}_g) = d(x_t, y) \leqslant r$. Since $r \leqslant \Delta(\mathcal{O}_g)/(2\operatorname{LIP}(g))$ then $d(g(x_t), g(y)) \leqslant \Delta(\mathcal{O}_g)/2$, thus $d(g(x_t), \mathcal{O}^g(y)) = d(g(x_t), \mathcal{O}_g) \leqslant r$. By induction it can then be shown that for all $n \in \mathbb{N}$, $d(g^n(x_t), g^n(y)) = d(g^n(x_t), \mathcal{O}_g) \leqslant r$. Now $r \leqslant \delta$, so

$$\begin{aligned}\liminf_{n\to+\infty} \frac{1}{n} S_n^g \psi_C(x_t) &\leqslant \liminf_{n\to+\infty} \frac{1}{n} S_n^g \psi_g(x_t) = \liminf_{n\to+\infty} \frac{1}{n} S_n^g \overline{\phi}(x_t) \\ &\leqslant \lim_{n\to+\infty} \frac{1}{n} S_n^g \overline{\phi}(y) + \liminf_{n\to+\infty} \frac{1}{n} |\phi|_\alpha \sum_{i=0}^{n-1} d\big(g^i(x_t), g^i(y)\big)^\alpha \\ &\leqslant \eta + \liminf_{n\to+\infty} \frac{2}{n} K |\phi|_\alpha \frac{\lambda^\alpha}{\lambda^\alpha - 1} \big(d(x_t, y) + d\big(g^{n-1}(x_t), g^{n-1}(y)\big)\big)^\alpha,\end{aligned}$$

therefore $\liminf_{n\to+\infty} \frac{1}{n} S_n^g \psi_C(x_t) \leqslant \eta + \liminf_{n\to+\infty} \frac{2K}{n} (2r)^\alpha |\phi|_\alpha \frac{\lambda^\alpha}{\lambda^\alpha - 1} = \eta$, so (2.49) follows.

The third case is where $\mathcal{O}(x_t)$ is not contained in $B(\mathcal{O}_g, r)$, but $x_t \in B(\mathcal{O}_g, \rho)$. Let $y \in \mathcal{O}_g$ with $d(x_t, y) = d(x_t, \mathcal{O}_g)$. Define $N := \min\big\{i \in \mathbb{N}_0 : d\big(g^{i+1}(x_t), g^{i+1}(y)\big) \geqslant r\big\}$ and $m := \max\big\{i \in \mathbb{N}_0 : i \leqslant N,\ d\big(g^{i-1}(x_t), g^{i-1}(y)\big) < \rho\big\}$ (cf. the similar definitions (2.35) and (2.36) for expanding maps), where $r \leqslant \Delta(\mathcal{O}_g)/(2\operatorname{LIP}(g))$ implies $d\big(g^i(x_t), \mathcal{O}_g\big) = d\big(g^i(x_t), g^i(y)\big)$ if $0 \leqslant i \leqslant N$. We define $n_t := N + 1$ and $x_{t+1} := g^{N+1}(x_t)$. Then $S_{n_t}^g \psi_C(x_t) \leqslant 0$ can be obtained via an argument analogous to the one used in the proof of Theorem 2.7, by separately estimating four terms, and choosing C sufficiently large. The only significant difference is that in the inequality (2.41), the term $\big|S_m^T \psi'(x_t) - S_m^T \psi'(y)\big|$ is estimated via the distance-expanding property, whereas here we estimate $\big|S_m^g \psi_g(x_t) - S_m^g \psi_g(y)\big|$ using (2.48). This proves (2.49), and completes the recursive step.

Having shown that $\mu_{\mathcal{O}_g}$ is (g, ψ_C)-maximizing, it follows that $\mu_{\mathcal{O}_g}$ is the *unique* g-maximizing measure for the function $\phi - 2C|\phi|_\alpha d_\infty(h_g, \mathrm{id})^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha$, as required. □

The above joint perturbation result allows us to deduce the following slightly stronger version of Theorem B:

Theorem B′ (Joint TPO for Anosov diffeomorphisms). *Let M be a smooth compact Riemannian manifold, with distance function induced by the Riemannian metric, and let $\mathcal{A}(M)$ be the space of C^1 Anosov diffeomorphisms on M, equipped with the C^1 topology. For all $\alpha \in (0,1]$, the set $\big\{(f,\phi) \in \mathcal{A}(M) \times C^{0,\alpha}(M) : \phi \in \operatorname{Lock}^\alpha(f)\big\}$ is open and dense in $\mathcal{A}(M) \times C^{0,\alpha}(M)$.*

Proof. This follows from Theorem 2.12 by an argument analogous to the one used to prove Theorem A′ from Theorem 2.7, and the fact that every Anosov diffeomorphism has TPO, by [HLMXZ19]. □

In fact Joint TPO can also be proved in the context of C^1 function spaces: if $C^1(M)$ is the space of C^1 functions on M, equipped with its usual topology, then there is an open dense subset of pairs $(T, \phi) \in \mathcal{A}(M) \times C^1(M)$ with the periodic optimization property. More specifically, if $\operatorname{Lock}^{C^1}(f)$ denotes the set of those $\phi \in C^1(M)$ whose f-maximizing measure is unique and periodic, with $\mathcal{M}_{\max}(f,\phi) = \mathcal{M}_{\max}(f,\psi)$ for all $\psi \in C^1(M)$ sufficiently close to ϕ in $C^1(M)$, then we have:

Theorem 2.13 (Joint TPO for Anosov diffeomorphisms, C^1 function space). *Let M be a smooth compact Riemannian manifold, with distance function induced by the Riemannian metric, and let $\mathcal{A}(M)$ be the space of C^1 Anosov diffeomorphisms on M, equipped with the C^1 topology. The set $\big\{(f,\phi) \in \mathcal{A}(M) \times C^1(M) : \phi \in \operatorname{Lock}^{C^1}(f)\big\}$ is open and dense in $\mathcal{A}(M) \times C^1(M)$.*

Proof. We will first establish a C^1 joint perturbation result. In other words, we will show that for $f \in \mathcal{A}(M)$, and $\mathcal{O}$ an f-periodic orbit, for all $g \in \mathcal{A}(M)$ sufficiently close to f, setting $\mathcal{O}_g := h_g(\mathcal{O})$, there exists a function $v_g \in C^1(M)$ such that (a) $\mathcal{M}_{\max}(g, v_g) = \{\mu_{\mathcal{O}_g}\}$, (b) for all $\phi \in C^1(M)$ with $\mathcal{M}_{\max}(f,\phi) = \{\mu_{\mathcal{O}}\}$, the periodic measure $\mu_{\mathcal{O}_g}$ is the unique $(g, \phi + |\phi|_1 v_g)$-maximizing measure, and (c) $\lim_{g\to f} \|\mathrm{D}v_g\|_\infty = 0$.

Let us fix $f \in \mathcal{A}(M)$, and the f-periodic orbit $\mathcal{O}$, and for $\alpha = 1$ let the neighbourhood U, and constant $C > 0$, be as in Theorem 2.12. By [HLMXZ19, Proposition 2.5 and (2.26)], choosing $u = \mathbb{0}$ to be the function that is identically zero, we see that there exists $\delta > 0$ such that for every $g \in U$, if $\xi \in C^{0,1}(M)$ with $|\xi|_1 < 3$ and $\|\xi\|_\infty < \delta$, then for $d_{g,\xi} := -d(\cdot, \mathcal{O}_g) + \xi$, the unique $(g, d_{g,\xi})$-maximizing measure is $\mu_{\mathcal{O}_g}$.

Now fix $g \in U$, and let $\phi \in C^1(M)$ with $\mathcal{M}_{\max}(f, \phi) = \{\mu_{\mathcal{O}}\}$. Since ϕ is C^1, and therefore Lipschitz, the case $\alpha = 1$ of Theorem 2.12 gives that $\mathcal{M}_{\max}(g, \phi - 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)) = \left\{\mu_{\mathcal{O}_g}\right\}$, where $C_g := Cd_\infty(h_g, \mathrm{id})^{1/2}$. By [HLMXZ19, Theorem 2.7], there exists $w \in C^1(M)$ with $\|\mathrm{D}w\|_\infty < 3/2$ and $\|w + d(\cdot, \mathcal{O}_g)\|_\infty < (5/6)\delta$. Define the C^1 function $\psi := \phi + 12C_g|\phi|_1 w$, and write

$$\psi = (\phi - 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)) + (12C_g|\phi|_1 w + 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)). \tag{2.50}$$

The above inequalities can be used to estimate

$$12C_g|\phi|_1|w + d(\cdot, \mathcal{O}_g)|_1 \leqslant 12C_g|\phi|_1(|w|_1 + |d(\cdot, \mathcal{O}_g)|_1) < 30|\phi|_1 C_g \quad \text{and}$$
$$12C_g|\phi|_1\|w + d(\cdot, \mathcal{O}_g)\|_\infty < 10C_g|\phi|_1\delta,$$

so taking $\xi := \frac{6}{5}(w + d(\cdot, \mathcal{O}_g))$ gives that $\mu_{\mathcal{O}_g}$ is the unique g-maximizing measure for the function $-d(\cdot, \mathcal{O}_g) + \frac{6}{5}(w + d(\cdot, \mathcal{O}_g))$, hence for its positive multiple $-10C_g|\phi|_1 d(\cdot, \mathcal{O}_g) + 12C_g|\phi|_1(w + d(\cdot, \mathcal{O}_g)) = 12C_g|\phi|_1 w + 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)$, and hence also for the function w. So both $\phi - 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)$ and $12C_g|\phi|_1 w + 2C_g|\phi|_1 d(\cdot, \mathcal{O}_g)$ have $\mu_{\mathcal{O}_g}$ as their unique g-maximizing measure, therefore by (2.50), $\mu_{\mathcal{O}_g}$ is the unique g-maximizing measure for ψ, and hence the desired joint perturbation result follows by taking $v_g := 12C_g w$.

The above C^1 joint perturbation result can then be used to prove Joint TPO, by an argument analogous to the one used in the proof of Theorem A′: more precisely, the ideas used in the proof of Theorem A′ can be adapted for the present C^1 case if the term $-Cd_{\mathrm{Lip}}(T_0, T)^{\alpha/2} d(\cdot, \mathcal{O})^\alpha = -Cd_{\mathrm{Lip}}(T_0, T)^{1/2} d(\cdot, \mathcal{O})$ is replaced by functions v_g satisfying conditions (a), (b), and (c), for g sufficiently close to f. □

3. Beta-transformations and maximizing measures

We now turn our attention to typical periodic optimization, and joint typical periodic optimization, for a specific one-parameter family of maps on the unit interval. This family of *beta-transformations* has been studied since the foundational papers of Rényi [Re57] and Parry [Pa60], motivated in particular by connections with aspects of number theory, in view of the link with *beta-expansions* of the form $\varepsilon_1/\beta + \varepsilon_2/\beta^2 + \varepsilon_3/\beta^3 + \cdots$ (see e.g. [AB07, Be86, CK04, DK02, DK03, FS92, Ka15, Sc80, Si03]). Beta-transformations have also been studied from the point of view of symbolic dynamics (see e.g. [AJ09, Bl89, IT74, LiSc05, Sc97, Si76]) and of ergodic theory (see e.g. [Hof78, Sm73, Wal78]).

The classical nature of this subject means that certain preparatory results in this section are either known, or resemble known results, though the literature is somewhat scattered; for ease of exposition, proofs are deferred until Appendix A. In Subsection 3.1 we recall the definitions of beta-transformations, beta-expansions, and beta-shifts, and the fundamental relations between these objects. Certain monotonicity and approximation properties as a function of the parameter β are considered in Subsection 3.2, along with notation and results concerning cylinder sets. In Subsection 3.3, we develop a theory of ergodic optimization for discontinuous maps such as beta-transformations, and relations between various sets of invariant measures are established.

3.1. Beta-transformations, beta-expansions, and beta-shifts.

We begin by recalling the definitions and basic properties of beta-transformations, as well as the related beta-expansions and beta-shifts.

Definition 3.1 (Beta-transformations). Given a real number $\beta > 1$, the *beta-transformation* $T_\beta \colon I \to I$ is defined by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor, \quad x \in I. \tag{3.1}$$

Recall that $\lfloor x \rfloor' = \max\{n \in \mathbb{Z} : n < x\}$ for $x \in \mathbb{R}$. The *upper beta-transformation* $U_\beta \colon I \to I$ is defined by $U_\beta(0) := 0$ and

$$U_\beta(x) := \beta x - \lfloor \beta x \rfloor', \quad x \in I \smallsetminus \{0\}. \tag{3.2}$$

Note that Kalle and Steiner [KS12, Definition 2.4] refer to the upper beta-transformation as the left-continuous beta-transformation.

Definition 3.2 (Beta-expansions). Given a real number $\beta > 1$, write

$$\mathcal{B} := \{0, 1, \dots, \lfloor \beta \rfloor\}.$$

Define the *β-expansion* of $x \in I$ to be the sequence

$$\underline{\varepsilon}(x, \beta) = \{\varepsilon_n(x, \beta)\}_{n \in \mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$$

given by

$$\varepsilon_n(x, \beta) := \left\lfloor \beta T_\beta^{n-1}(x) \right\rfloor \quad \text{for all } n \in \mathbb{N}, \tag{3.3}$$

and define the *upper β-expansion*[24] of $x \in I$ to be the sequence

$$\underline{\varepsilon}^*(x, \beta) = \{\varepsilon_n^*(x, \beta)\}_{n \in \mathbb{N}} \in \mathcal{B}^{\mathbb{N}}$$

given by

$$\varepsilon_n^*(x, \beta) := \left\lfloor \beta U_\beta^{n-1}(x) \right\rfloor' \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Remark 3.3. For $\beta > 1$, the beta-transformation and upper beta-transformation are related by

$$U_\beta(x) = \limsup_{y \to x} T_\beta(y).$$

The set D_β of points of discontinuity for T_β is

$$D_\beta := T_\beta^{-1}(0) \smallsetminus \{0\} = U_\beta^{-1}(1) = \{j/\beta : j \in \mathbb{Z}\} \cap (0, 1], \tag{3.5}$$

and this is precisely the set of points at which T_β and U_β differ, with $T_\beta(x) = 0$ and $U_\beta(x) = 1$ for all $x \in D_\beta$.

Lemma 3.4. *If $\beta > 1$, $n \in \mathbb{N}$, $a \in [0, 1)$, and $b \in (0, 1]$, then:*

(i) $\lim_{x \searrow a} T_\beta^n(x) = T_\beta^n(a)^+$ *and* $\lim_{x \nearrow b} U_\beta^n(x) = U_\beta^n(b)^-$.

(ii) $\varepsilon_n(\cdot, \beta)$ *is right-continuous on* $[0, 1)$ *and* $\varepsilon_n^*(\cdot, \beta)$ *is left-continuous on* $(0, 1]$.

(iii) $T_\beta^n(0) = \varepsilon_n(0, \beta) = U_\beta^n(0) = \varepsilon_n^*(0, \beta) = 0$.

(iv) $\lim_{x \nearrow b} T_\beta^n(x) = U_\beta^n(b)^-$ *and* $\lim_{x \nearrow b} \varepsilon_n(x, \beta) = \varepsilon_n^*(b, \beta)$.

Lemma 3.5. *If $\beta > 1$, $n \in \mathbb{N}$, and $x \in I$, then:*

(i) $\lim_{\gamma \searrow \beta} T_\gamma^n(x) = T_\beta^n(x)^+$ *and* $\lim_{\gamma \nearrow \beta} U_\beta^n(x) = U_\beta^n(x)^-$.

(ii) $\varepsilon_n(x, \cdot)$ *is right-continuous and* $\varepsilon_n^*(x, \cdot)$ *is left-continuous.*

(iii) $\lim_{\gamma \nearrow \beta} T_\gamma^n(x) = U_\beta^n(x)^-$ *and* $\lim_{\gamma \nearrow \beta} \varepsilon_n(x, \gamma) = \varepsilon_n^*(x, \beta)$.

Definition 3.6 (Beta-shifts). Given $\beta > 1$, define $\pi_\beta \colon I \to \mathcal{B}^{\mathbb{N}}$ by

$$\pi_\beta(x) := \underline{\varepsilon}(x, \beta) = \{\varepsilon_n(x, \beta)\}_{n \in \mathbb{N}},$$

and define $\pi_\beta^* \colon I \to \mathcal{B}^{\mathbb{N}}$ by

$$\pi_\beta^*(x) := \underline{\varepsilon}^*(x, \beta) = \{\varepsilon_n^*(x, \beta)\}_{n \in \mathbb{N}}.$$

Define the *beta-shift* $\mathcal{S}_\beta$ to be the closure in $\mathcal{B}^{\mathbb{N}}$ of the image under π_β of the half-open interval $[0, 1)$, in other words,

$$\mathcal{S}_\beta := \overline{\pi_\beta([0, 1))}, \tag{3.6}$$

where $\mathcal{B}^{\mathbb{N}}$ is equipped with the product topology. Note that Lemma 3.5 (iii) implies that $\pi_\beta^*(I) \subseteq \mathcal{S}_\beta$.

[24] Blanchard [Bl89, p. 136] refers to the upper β-expansion $\underline{\varepsilon}^*(x, \beta)$ as a kind of *incorrect* β-expansion.

Definition 3.7. Given $\beta > 1$, let X_β denote the closure in $\mathcal{B}^{\mathbb{N}}$ of the image $\pi_\beta(I)$. Define $h_\beta \colon X_\beta \to I$ by

$$h_\beta(\{z_i\}_{i\in\mathbb{N}}) \coloneqq \sum_{i=1}^{+\infty} z_i \beta^{-i}. \tag{3.7}$$

For each $x \in I$, define $i_x \colon (1, +\infty) \to \mathbb{N}_0^{\mathbb{N}}$ and $i_x^* \colon (1, +\infty) \to \mathbb{N}_0^{\mathbb{N}}$ by

$$i_x(\beta) \coloneqq \pi_\beta(x) \quad \text{and} \quad i_x^*(\beta) \coloneqq \pi_\beta^*(x). \tag{3.8}$$

The following lemma means that our definition of upper β-expansion is equivalent to the definition of incorrect β-expansion in [IT74] and [YT21].

Lemma 3.8. *Fix $\beta > 1$. Then $\pi_\beta(0) = \pi_\beta^*(0) = (0)^\infty$ and $\pi_\beta^*(a) = \lim_{x\nearrow a} \pi_\beta(x)$ for all $a \in (0, 1]$.*

Proposition 3.9 below collects a number of basic properties of beta-transformations and beta-expansions that will be required later; the majority of the results can be found in the existing literature (specifically, in [Bl89, IT74, Pa60, Re57, YT21]), and for the remainder we provide proofs.

Proposition 3.9. *If $\beta > 1$, then:*

(i) *We have*

$$\pi_\beta^*(1) = \begin{cases} (z_1 z_2 \dots (z_n - 1))^\infty & \text{if } \pi_\beta(1) = z_1 z_2 \dots z_n (0)^\infty, z_n > 0, \\ \pi_\beta(1) & \text{if } \pi_\beta(1) \text{ has infinitely many nonzero terms.} \end{cases}$$

(ii) *For each $x \in (0, 1]$,*

$$\pi_\beta^*(x) = \begin{cases} z_1 z_2 \dots (z_n - 1)\pi_\beta^*(1) & \text{if } \pi_\beta(x) = z_1 z_2 \dots z_n (0)^\infty, z_n > 0, \\ \pi_\beta(x) & \text{if } \pi_\beta(x) \text{ has infinitely many nonzero terms.} \end{cases}$$

(iii) *$\sigma \circ \pi_\beta = \pi_\beta \circ T_\beta$ and $\sigma \circ \pi_\beta^* = \pi_\beta^* \circ U_\beta$ on I.*

(iv) *$(h_\beta \circ \pi_\beta)(x) = x$ and $(h_\beta \circ \pi_\beta^*)(x) = x$ for each $x \in I$.*

(v) *$h_\beta \circ \sigma = T_\beta \circ h_\beta$ on $\pi_\beta(I)$ and $h_\beta \circ \sigma = U_\beta \circ h_\beta$ on $\pi_\beta^*(I)$.*

(vi) *π_β and π_β^* are strictly increasing, i.e., $x < y$ implies $\pi_\beta(x) \prec \pi_\beta(y)$ and $\pi_\beta^*(x) \prec \pi_\beta^*(y)$.*

(vii) *$\pi_\beta(x) \prec \pi_\beta^*(y)$ if $0 \leqslant x < y \leqslant 1$.*

(viii) *$\{\omega \in X_\beta : \pi_\beta^*(x) \prec \omega \prec \pi_\beta(x)\} = \emptyset$ for all $x \in I$.*

(ix) *π_β is right-continuous on $[0, 1)$ and π_β^* is left-continuous on $(0, 1]$.*

(x) *h_β is a continuous surjection and is non-decreasing, i.e., $\omega \prec \omega'$ implies $h_\beta(\omega) \leqslant h_\beta(\omega')$.*

(xi) *The inverse image $h_\beta^{-1}(x)$ of $x \in (0, 1]$ consists either of one point $\pi_\beta(x)$ or of two distinct points $\pi_\beta(x)$ and $\pi_\beta^*(x)$. The latter case occurs only when $T_\beta^n(x) = 0$ for some $n \in \mathbb{N}$. Moreover, $h_\beta^{-1}(0) = \{(0)^\infty\}$.*

(xii) *The function $h_\beta \colon (X_\beta, d_\beta) \to (I, d)$ is Lipschitz.*

(xiii) *For each $x \in (0, 1]$, the functions i_x and i_x^* are both strictly increasing functions. Moreover, $i_0(\beta) = i_0^*(\beta) = (0)^\infty$ for all $\beta > 1$.*

(xiv) *For each $x \in I$, the function i_x is right-continuous and the function i_x^* is left-continuous.*

The following classification of values $\beta > 1$, and the interpretation in terms of dynamical behaviour, will be required in our subsequent investigations.

Definition 3.10 (Classification of $\beta > 1$). A real number $\beta > 1$ is said to be

(i) a *simple beta-number* if $\underline{\varepsilon}(1, \beta)$ has only finitely many nonzero terms;

(ii) a *non-simple beta-number* if $\underline{\varepsilon}(1, \beta)$ is preperiodic (i.e., there exists $n \in \mathbb{N}$ such that $\sigma^n(\underline{\varepsilon}(1, \beta))$ is periodic), but β is not a simple beta-number;

(iii) *non-preperiodic* if β is not a beta-number (i.e., β satisfies neither (i) nor (ii) above).

Remark 3.11. The terminology *beta-number*, as well as *simple beta-number*, was introduced by Parry [Pa60], who proved (see [Pa60, Theorem 5]) that the set of simple beta-numbers is dense in $(1, +\infty)$. Some authors refer to simple beta-numbers as *Parry numbers* (see e.g. [Ka15]). It is readily seen that β is a simple beta-number if and only if 1 is a periodic point of U_β.

The following proposition summarises the relation between periodic points and invariant measures of T_β and U_β.

Proposition 3.12. *If $\beta > 1$, then:*

(i) $T_\beta^{-1}(0) = \{0\} \cup D_\beta$, *where* $\{0\} \cap D_\beta = \emptyset$. *Moreover,* $T_\beta^{-1}(1) = \emptyset$, $U_\beta^{-1}(0) = \{0\}$, *and* $U_\beta^{-1}(1) = D_\beta$. *The maps T_β and U_β coincide when restricted to $I \smallsetminus D_\beta$.*

(ii) $\operatorname{Per}(T_\beta) \subseteq \operatorname{Per}(U_\beta)$. *If $\mathcal{O}_\beta^*$ is a periodic orbit for U_β, then $\mathcal{O}_\beta^*$ is a periodic orbit for T_β if and only if $1 \notin \mathcal{O}_\beta^*$.*

(iii) $\mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$. *If $\mu \in \mathcal{M}(I, U_\beta)$, then $\mu \in \mathcal{M}(I, T_\beta)$ if and only if $\mu(\{1\}) = 0$.*

(iv) *If β is not a simple beta-number, then* $\operatorname{Per}(T_\beta) = \operatorname{Per}(U_\beta)$ *and* $\mathcal{M}(I, T_\beta) = \mathcal{M}(I, U_\beta)$.

(v) *If β is a simple beta-number, then* $\operatorname{Per}(U_\beta) = \operatorname{Per}(T_\beta) \cup \mathcal{O}_\beta^*(1)$ *and $\mathcal{M}(I, U_\beta)$ is the convex hull of* $\big\{\mu_{\mathcal{O}_\beta^*(1)}\big\} \cup \mathcal{M}(I, T_\beta)$.

(vi) *T_β and U_β are distance-expanding. Specifically, if $x, y \in I$ with $|x - y| < 1/(2\beta)$, then* $|T_\beta(x) - T_\beta(y)| \geqslant \beta|x - y|$ *and* $|U_\beta(x) - U_\beta(y)| \geqslant \beta|x - y|$.

While the support of any T-invariant probability measure μ satisfies $T(\operatorname{supp}\mu) = \operatorname{supp}\mu$ in the case where T is continuous (see e.g. [Ak93, p. 156]), the same is not true for the discontinuous maps T_β and U_β; nevertheless we do have the following result.

Lemma 3.13. *Suppose $\beta > 1$ and $\mu \in \mathcal{M}(I, U_\beta)$. Then* $U_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$ *if* $0 \notin \operatorname{supp}\mu$, *and* $T_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$ *if* $1 \notin \operatorname{supp}\mu$.

3.2. **Monotonicity and approximation properties in parameter space.** Here we recall some monotonicity and approximation properties for the one-parameter family of beta-shifts.

The following proposition characterises those sequences on the alphabet $\mathcal{B} = \{0, 1, \dots, \lfloor\beta\rfloor\}$ that arise as the β-expansion of a real number $x \in [0, 1]$.

Proposition 3.14. *If $\beta > 1$, then:*

(i) $\pi_\beta([0,1)) = \big\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \prec \pi_\beta^*(1)$ *for all* $n \in \mathbb{N}_0\big\}$.

(ii) *$\mathcal{S}_\beta$ can also be expressed as*

$$\mathcal{S}_\beta = \big\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \preceq \pi_\beta^*(1) \text{ for all } n \in \mathbb{N}_0\big\}. \tag{3.9}$$

The representation (3.9) implies that the closed subset $\mathcal{S}_\beta$ satisfies $\sigma(\mathcal{S}_\beta) = \mathcal{S}_\beta$; in other words it is a subshift of the full shift $\mathcal{B}^{\mathbb{N}}$, so we may regard the shift map σ as a self-map $\sigma\colon \mathcal{S}_\beta \to \mathcal{S}_\beta$.

Remark 3.15. (i) Define $\widetilde{X}_\beta \coloneqq \big\{A \in \mathcal{B}^{\mathbb{N}} : \sigma^n(A) \preceq \pi_\beta(1) \text{ for all } n \in \mathbb{N}_0\big\}$.

(ii) If β is not a simple beta-number, then $\pi_\beta(1) = \pi_\beta^*(1)$ (see Proposition 3.9 (i)), and hence $\mathcal{S}_\beta = X_\beta = \widetilde{X}_\beta$ by (3.9), Definitions 3.6 and 3.7.

(iii) If β is a simple beta-number, then $\pi_\beta^*(1) \prec \pi_\beta(1)$ (see Proposition 3.9 (i)), hence X_β is the union of the beta-shift $\mathcal{S}_\beta$ and the singleton set $\{\pi_\beta(1)\}$ (which is disjoint from $\mathcal{S}_\beta$), and

$$\mathcal{S}_\beta \subseteq X_\beta \subseteq \widetilde{X}_\beta \tag{3.10}$$

by (3.9), Definitions 3.6 and 3.7. The inclusions in (3.10) are proper: for example, when $\beta = 2$, we have

$$2(0)^\infty \in X_2 \smallsetminus \mathcal{S}_2, \quad 12(0)^\infty \in \widetilde{X}_2 \smallsetminus X_2.$$

(iv) If β is a simple beta-number then σ maps $\mathcal{S}_\beta$ surjectively onto itself, and maps $\widetilde{X}_\beta$ surjectively onto itself, but $\sigma\colon X_\beta \to X_\beta$ is not surjective.

(v) A complement to Proposition 3.14 (i) is that, for each $A \in \mathbb{N}_0^{\mathbb{N}}$, there exists $\beta > 1$ with $A = \underline{\varepsilon}(1, \beta)$ if and only if $\sigma^n(A) \prec A$ for all $n \in \mathbb{N}$; and if such a number $\beta > 1$ exists then it is unique (see [Pa60, Corollary 1]). Consequently, each of the three classes in Definition 3.10 is readily seen to be non-empty.

(vi) Some authors define the beta-shift to be either X_β or $\widetilde{X}_\beta$, instead of $\mathcal{S}_\beta$. For example it is defined to be X_β in [AB07, p. 1696], [Sc97, Definition 2.2], and [KQ22, p. 1438]), and defined to be $\widetilde{X}_\beta$ in [Si76, p. 248] and [Wal82, p. 179].

Lemma 3.16. *If $\beta > 1$, then:*

(i) *If $1 < \beta' < \beta$, then $\mathcal{S}_{\beta'} \subseteq \mathcal{S}_\beta$.*

(ii) $\mathcal{S}_\beta = \overline{\bigcup_{\gamma\in(1,\beta)} \mathcal{S}_\gamma}$.

Remark 3.17. Lemma 3.16 is hinted at as part of [IT74, Proposition 4.1] (though in [IT74] it is slightly mis-stated, and not proved, so for the convenience of the reader we include a proof in Appendix A). We note that another part of [IT74, Proposition 4.1] is false: in general it is not the case that $\mathcal{S}_\beta = \bigcap_{\gamma>\beta} \mathcal{S}_\gamma$ (for example if $\beta = 2$ then $2(0)^\infty \in \mathcal{S}_\gamma$ for all $\gamma > 2$, but $2(0)^\infty \notin \mathcal{S}_2$), however the intersection can be expressed as

$$\widetilde{X}_\beta = \bigcap_{\gamma>\beta} \mathcal{S}_\gamma.$$

Definition 3.18. For $1 < \gamma < \beta$, define

$$H_\beta^\gamma \coloneqq h_\beta(\mathcal{S}_\gamma) = \left\{ \sum_{i=1}^{+\infty} z_i \beta^{-i} : \{z_i\}_{i\in\mathbb{N}} \in \mathcal{S}_\gamma \right\}, \tag{3.11}$$

and if $\psi \in C^{0,\alpha}(I)$ then define the corresponding *restricted ergodic supremum*

$$Q_{\beta,\gamma}(\psi) \coloneqq Q\big(T_\beta|_{H_\beta^\gamma}, \psi|_{H_\beta^\gamma}\big) = \sup\left\{ \int \psi \,\mathrm{d}\mu : \mu \in \mathcal{M}(I, T_\beta),\, \operatorname{supp}\mu \subseteq H_\beta^\gamma \right\}. \tag{3.12}$$

Lemma 3.19. *Suppose $\beta > 1$. If $\mathcal{K} \subseteq I$ is a non-empty compact set with $1 \notin \mathcal{K} = T_\beta(\mathcal{K})$, then there exists $\beta' \in (1, \beta)$ such that $\mathcal{K} \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$.*

Similarly, if $\mathcal{K}^*$ is a non-empty compact set with $1 \notin \mathcal{K}^* = U_\beta(\mathcal{K}^*)$, then there exists $\beta' \in (1, \beta)$ such that $\mathcal{K}^* \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$.

Recall that a homeomorphism $g\colon Y_1 \to Y_2$ between metric spaces (Y_1, d_1) and (Y_2, d_2) is *bi-Lipschitz* if there exists a constant $C \geqslant 1$ such that for all $u, v \in Y_1$,

$$C^{-1} d_1(u, v) \leqslant d_2(g(u), g(v)) \leqslant C d_1(u, v).$$

Lemma 3.20. *For $1 < \gamma < \beta$, the map $\pi_\beta|_{H_\beta^\gamma}\colon \big(H_\beta^\gamma, d\big) \to (\mathcal{S}_\gamma, d_\beta)$ is bi-Lipschitz.*

Lemma 3.21. *For each $\beta > 1$ and each $\gamma \in (1, \beta)$, the set H_β^γ is a closed subset of I satisfying $T_\beta\big(H_\beta^\gamma\big) \subseteq H_\beta^\gamma$ and the restricted beta-transformation $T_\beta|_{H_\beta^\gamma}\colon H_\beta^\gamma \to H_\beta^\gamma$ has the following properties:*

(i) *$T_\beta|_{H_\beta^\gamma}$ is Lipschitz.*

(ii) *$T_\beta|_{H_\beta^\gamma}$ is distance-expanding.*

(iii) *If γ is a simple beta-number, then $T_\beta|_{H_\beta^\gamma}$ is an open mapping.*

We conclude this section with the following notation and results concerning cylinder subsets of I.

Definition 3.22. Fix $\beta > 1$ and $n \in \mathbb{N}$. A length-n prefix $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is said to be *β-admissible* if $\varepsilon_1 \dots \varepsilon_n (0)^\infty \in \pi_\beta([0,1))$.

For each β-admissible length-n prefix, we define the corresponding *n-cylinder* to be

$$I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \{x \in [0,1) : \varepsilon_i(x,\beta) = \varepsilon_i \text{ for all } 1 \leqslant i \leqslant n\}, \tag{3.13}$$

and if $T_\beta^n(I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)) = [0,1)$ we say that the cylinder $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is *full*. Let W^n denote the set of all n-cylinders, and let W_0^n denote the set of all full n-cylinders.

Note that the n-cylinder $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is a left-closed and right-open interval, with the left endpoint

$$\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n}. \tag{3.14}$$

Proposition 3.23. *Fix $\beta > 1$, $n \in \mathbb{N}$, and $I^n \coloneqq I(\varepsilon_1, \dots, \varepsilon_n) \in W^n$.*

(i) $[0,1) = \bigcup_{J^n \in W^n} J^n$, *and the n-cylinders J^n in W^n are pairwise disjoint.*
(ii) *If $m \in \{1, \dots, n\}$ and $x, y \in I^n$, then $T_\beta^m(y) - T_\beta^m(x) = \beta^m(y-x)$. Consequently, $T_\beta^m|_{I^n}$ is continuous and strictly increasing.*
(iii) *If $\varepsilon_n > 0$, then $I(\varepsilon_1, \dots, \varepsilon_{n-1}, b) \in W_0^n$ for $b \in \{0, \dots, \varepsilon_n - 1\}$ with the right endpoint $(b+1)\beta^{-n} + \sum_{i=1}^{n-1} \varepsilon_i \beta^{-i}$.*
(iv) *If $I^n \in W_0^n$ and the right endpoint of I^n is not 1, then there exists a T_β^n-fixed point in I^n.*
(v) *There exists $m \in \{0, 1, \dots, n\}$ such that $T_\beta^n(I^n) = [0, U_\beta^m(1))$.*

Lemma 3.24. *Fix $\beta > 1$. Suppose $\alpha \in (0,1]$ and $\phi \in C^{0,\alpha}(I)$. For all $n \in \mathbb{N}$, $I^n \in W^n$, and $x, y \in I^n$, we have*

$$|S_n\phi(x) - S_n\phi(y)| \leqslant \frac{|\phi|_\alpha}{\beta^\alpha - 1} \bigl|T_\beta^n(x) - T_\beta^n(y)\bigr|^\alpha.$$

3.3. **Maximizing measures.** Here we introduce the notion of *limit-maximizing measure*, which will be useful for a dynamical system, such as T_β, whose set of invariant measures is not necessarily weak* compact. For $\beta > 1$ and $\phi \in C(I)$, we first show that the existence of a maximizing measure for (I, U_β, ϕ) is equivalent to the existence of a maximizing measure for $(X_\beta, \sigma, \phi \circ h_\beta)$. We then prove that a measure is limit-maximizing for (I, T_β, ϕ) if and only if it is maximizing for (I, U_β, ϕ).

Definition 3.25. Let $T\colon X \to X$ be a Borel measurable map on a compact metric space X. For a Borel measurable function $\psi\colon X \to \mathbb{R}$, a probability measure μ is called a *(T,ψ)-limit-maximizing measure*, or simply a *ψ-limit-maximizing measure*, if it is a weak* accumulation point of $\mathcal{M}(X,T)$ and $\int \psi \,\mathrm{d}\mu = Q(T,\psi)$. We denote the set of (T,ψ)-limit-maximizing measures by $\mathcal{M}_{\max}^*(T,\psi)$.

Clearly, $\mathcal{M}_{\max}(T,\psi) \subseteq \mathcal{M}_{\max}^*(T,\psi)$. For $\beta > 1$, let us write

$$Z_\beta \coloneqq \bigl\{x \in I : \pi_\beta(x) \neq \pi_\beta^*(x)\bigr\}. \tag{3.15}$$

The following lemma collects together some basic properties of Z_β.

Lemma 3.26. *If $\beta > 1$, then:*

(i) $Z_\beta = \bigl(\bigcup_{n\in\mathbb{N}} T_\beta^{-n}(0)\bigr) \smallsetminus \{0\} = \bigcup_{n\in\mathbb{N}} U_\beta^{-n}(1)$ *and in particular $D_\beta \subseteq Z_\beta$.*
(ii) $h_\beta^{-1}(W) = \pi_\beta^*(W) \cup \pi_\beta(W \cap Z_\beta)$ *for each $W \subseteq I$ and $\pi_\beta(Z_\beta) \cap \pi_\beta^*(I) = \emptyset$.*
(iii) *If $x \in I \smallsetminus Z_\beta$, then $T_\beta^n(x) = U_\beta^n(x)$ for all $n \in \mathbb{N}$.*
(iv) *π_β and π_β^* are continuous on $I \smallsetminus Z_\beta$.*
(v) *$\mu(\pi_\beta(Z_\beta)) = 0$ for all $\mu \in \mathcal{M}(X_\beta, \sigma)$.*

Now we consider the relation between $\mathcal{M}(I, U_\beta)$ and $\mathcal{M}(X_\beta, \sigma)$.

Notation. Fix $\beta > 1$. Define
$$G_\beta \colon \mathcal{M}(I, U_\beta) \to \mathcal{M}(X_\beta, \sigma)$$
to be the pushforward of π_β^*, in other words,
$$G_\beta(\mu)(Y) := \mu\big(\big(\pi_\beta^*\big)^{-1}(Y)\big) \tag{3.16}$$
for each $\mu \in \mathcal{M}(I, U_\beta)$ and each Borel measurable subset $Y \subseteq X_\beta$. By Proposition 3.9 (iii), it is straightforward to check that G_β is well-defined. Define
$$H_\beta \colon \mathcal{M}(X_\beta, \sigma) \to \mathcal{P}(I)$$
by
$$H_\beta(\nu)(W) := \nu\big(h_\beta{}^{-1}(W)\big) \tag{3.17}$$
for each $\nu \in \mathcal{M}(X_\beta, \sigma)$ and each Borel measurable subset $W \subseteq I$.

Proposition 3.27. *If $\beta > 1$ and $\phi \in C(I)$, then:*

(i) *H_β is a homeomorphism from $\mathcal{M}(X_\beta, \sigma)$ to $\mathcal{M}(I, U_\beta)$ with respect to the weak* topology, with $G_\beta^{-1} = H_\beta$.*
(ii) *$\mathcal{M}(I, U_\beta)$ is compact in the weak* topology.*
(iii) *$Q(U_\beta, \phi) = Q(\sigma|_{X_\beta}, \phi \circ h_\beta)$ and $\mathcal{M}_{\max}(U_\beta, \phi) = H_\beta(\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi \circ h_\beta)) \neq \emptyset$.*
(iv) *If $\mathcal{O}$ is an (X_β, σ)-periodic orbit, then $h_\beta(\mathcal{O})$ is an (I, U_β)-periodic orbit, with $\operatorname{card} \mathcal{O} = \operatorname{card} h_\beta(\mathcal{O})$.*

Proposition 3.28. *If $\beta > 1$ and $\phi \in C(I)$, then:*

(i) *$\mathcal{M}(I, U_\beta)$ is equal to the weak* closure of $\mathcal{M}(I, T_\beta)$.*
(ii) *$Q(T_\beta, \phi) = Q(U_\beta, \phi)$.*
(iii) *$\mathcal{M}^*_{\max}(T_\beta, \phi) = \mathcal{M}_{\max}(U_\beta, \phi)$.*

4. A Mañé cohomology lemma for beta-transformations

The purpose of this section is to prove a version of the *Mañé cohomology lemma* (Theorem 4.9) in the context of beta-transformations, and derive a new *revelation theorem* (Theorem 4.12), an important consequence regarding the support of a maximizing measure. To establish this result, there are several differences and difficulties compared to the Mañé lemma for open expanding maps (Theorem 2.6), notably the lack of continuity and openness of beta-transformations, so the method of proof here will be rather different (see Remark 4.11 for further details). A key tool is the introduction of an operator analogous to the one used by Bousch [Bou00], and show (Proposition 4.7) that it has a fixed point function with certain regularity properties (following [GLT09], this fixed point can be referred to as a *calibrated sub-action*).

For a Borel measurable map $T \colon I \to I$, and bounded Borel measurable function $\psi \colon I \to \mathbb{R}$, to study the (T, ψ)-maximizing measures it is convenient, whenever possible, to consider a cohomologous function $\widetilde{\psi}$ satisfying $\widetilde{\psi} \leqslant Q(T, \psi)$. We recall the following (cf. [Je19, p. 2601]):

Definition 4.1. Suppose $T \colon I \to I$ is Borel measurable, and $\psi \colon I \to \mathbb{R}$ is bounded and Borel measurable. If $\psi \leqslant Q(T, \psi)$ and $\psi^{-1}(Q(T, \psi))$ contains $\operatorname{supp} \mu$ for some $\mu \in \mathcal{M}(I, T)$, then ψ is said to be *revealed*. If $Q(T, \psi) = 0$ then ψ is said to be *normalised*; in particular, a normalised function ψ is revealed if and only if $\psi \leqslant 0$ and $\psi^{-1}(0)$ contains $\operatorname{supp} \mu$ for some $\mu \in \mathcal{M}(I, T)$.

Lemma 4.2. *Suppose $T \colon I \to I$ is Borel measurable, $\phi \colon I \to \mathbb{R}$ is bounded and Borel measurable, and $\mathcal{M}_{\max}(T, \phi) \neq \emptyset$. Denote $\overline{\phi} = \phi - Q(T, \phi)$, and suppose $\widetilde{\phi} = \overline{\phi} + u - u \circ T$ for some bounded Borel measurable function $u \colon I \to \mathbb{R}$. Then:*

(i) *$Q\big(T, \widetilde{\phi}\big) = Q\big(T, \overline{\phi}\big) = 0$.*
(ii) *$\mathcal{M}_{\max}(T, \phi) = \mathcal{M}_{\max}\big(T, \overline{\phi}\big) = \mathcal{M}_{\max}\big(T, \widetilde{\phi}\big)$.*

(iii) *If* $\widetilde{\phi} \leqslant 0$ *and if* $x \in I$ *is such that* $\mathcal{O}^T(x) \subseteq \widetilde{\phi}^{-1}(0)$*, then* $\mathcal{O}^T(x)$ *is a* (T, ϕ)*-maximizing orbit.*

Proof. (i) and (ii) follow from (1.3), (1.4), and the fact that

$$\int \widetilde{\phi}\, \mathrm{d}\mu = \int \left(\overline{\phi} + u - u \circ T\right) \mathrm{d}\mu = \int \overline{\phi}\, \mathrm{d}\mu \quad \text{ for all } \mu \in \mathcal{M}(I, T).$$

If $\widetilde{\phi} \leqslant 0$ and $\mathcal{O}^T(x) \subseteq \widetilde{\phi}^{-1}(0)$, then $0 = \frac{1}{n} S_n^T \widetilde{\phi}(x) = \frac{1}{n} S_n^T \overline{\phi}(x) + \frac{1}{n}(u(x) - u(T^n(x)))$ for all $n \in \mathbb{N}$, and (iii) follows from the fact that u is bounded. □

The following operator[25] $\mathcal{L}_\psi$ is an analogue of the one used by Bousch in [Bou00].

Definition 4.3. Let $\psi \colon [0, 1) \to \mathbb{R}$ be bounded and Borel measurable. For $\beta > 1$, define $\mathcal{L}_\psi \colon \mathbb{R}^{[0,1)} \to \mathbb{R}^{[0,1)}$ by

$$\mathcal{L}_\psi(u)(x) \coloneqq \max\bigl\{(u + \psi)(y) : y \in T_\beta^{-1}(x) \smallsetminus \{1\}\bigr\}, \quad x \in [0, 1). \tag{4.1}$$

Note that $\mathcal{L}_\psi$ is well defined since $T_\beta([0, 1)) = [0, 1)$ (cf. (3.1)), and if $u \colon [0, 1) \to \mathbb{R}$ is bounded then so is $\mathcal{L}_\psi(u)$. For a function $u \colon I \to \mathbb{R}$ and a bounded Borel measurable function $\psi \colon I \to \mathbb{R}$, we define $\mathcal{L}_\psi(u) \coloneqq \mathcal{L}_{\psi|_{[0,1)}}\bigl(u|_{[0,1)}\bigr)$.

Lemma 4.4. *If* $\beta > 1$ *and* $\psi \colon I \to \mathbb{R}$ *is bounded and Borel measurable, and* $\overline{\psi} \coloneqq \psi - Q(T_\beta, \psi)$*, then:*

(i) *If* $x \in [0, 1)$*,* $n \in \mathbb{N}$*, and* $u \colon I \to \mathbb{R}$ *is bounded, then*

$$\mathcal{L}_{\overline{\psi}}^n(u)(x) + nQ(T_\beta, \psi) = \mathcal{L}_\psi^n(u)(x) = \max\bigl\{u(y) + S_n\psi(y) : y \in T_\beta^{-n}(x) \smallsetminus \{1\}\bigr\}.$$

(ii) $\mathcal{L}_\psi(\sup_{v \in \mathcal{A}} v) = \sup_{v \in \mathcal{A}} \mathcal{L}_\psi(v)$ *for any collection* $\mathcal{A}$ *of bounded real-valued functions on* $[0, 1)$*.*

(iii) *If* $\{u_n\}_{n \in \mathbb{N}}$ *is a pointwise convergent sequence of bounded real-valued functions on* $[0, 1)$*, then* $\lim\limits_{n \to +\infty} \mathcal{L}_\psi(u_n) = \mathcal{L}_\psi\bigl(\lim\limits_{n \to +\infty} u_n\bigr)$*, where* $\lim\limits_{n \to +\infty}$ *denotes pointwise limit.*

Proof. (i) The first equality in (i) is immediate from (4.1) and the fact that $\overline{\psi} = \psi - Q(T_\beta, \psi)$. As $T_\beta([0, 1)) = [0, 1)$, then

$$\max\bigl\{u(y) + S_n\psi(y) : y \in T_\beta^{-n}(x) \smallsetminus \{1\}\bigr\} = \max\bigl\{u(y) + S_n\psi(y) : y \in \bigl(T_\beta|_{[0,1)}\bigr)^{-n}(x)\bigr\} \tag{4.2}$$

for all $n \in \mathbb{N}$ and $x \in [0, 1)$. Then the second equality is easily proved by (4.1), (4.2), and induction (cf. e.g. [JMU06, JMU07]).

(ii) follows readily from the fact that $\psi + \sup_{v \in \mathcal{A}} v = \sup_{v \in A}(\psi + v)$, and that for each $x \in [0, 1)$,

$$\max_{y = T_\beta^{-1}(x) \smallsetminus \{1\}} \sup_{v \in \mathcal{A}} (\psi + v)(y) = \sup_{v \in \mathcal{A}} \max_{y = T_\beta^{-1}(x) \smallsetminus \{1\}} (\psi + v)(y).$$

(iii) Define $v \colon [0, 1) \to \mathbb{R}$ by $v(x) \coloneqq \lim_{n \to +\infty} u_n(x)$ for all $x \in [0, 1)$. Fix arbitrary $x \in [0, 1)$ and $\varepsilon > 0$. Then there exists $N = N(x, \varepsilon) \in \mathbb{N}$ such that if $n \geqslant N$ then $|u_n(y) - v(y)| < \varepsilon$ for each of the finitely many pre-images $y \in T_\beta^{-1}(x)$.

Fix $n \geqslant N$. Let y_1, $y_2 \in T_\beta^{-1}(x) \smallsetminus \{1\}$ satisfy $\mathcal{L}_\psi(u_n)(x) = (\psi + u_n)(y_1)$ and $\mathcal{L}_\psi(v)(x) = (\psi + v)(y_2)$, so that

$$\bigl(\mathcal{L}_\psi(u_n) - \mathcal{L}_\psi(v)\bigr)(x) \leqslant (\psi + u_n)(y_1) - (\psi + v)(y_1) = u_n(y_1) - v(y_1) < \varepsilon \quad \text{ and}$$
$$\bigl(\mathcal{L}_\psi(u_n) - \mathcal{L}_\psi(v)\bigr)(x) \geqslant (\psi + u_n)(y_2) - (\psi + v)(y_2) = u_n(y_2) - v(y_2) > -\varepsilon.$$

Then (iii) follows. □

Notation. For $\beta > 1$ and $\alpha \in (0, 1]$, we write

$$K_{\alpha,\beta} \coloneqq \frac{1}{\beta^\alpha - 1}. \tag{4.3}$$

[25] Although non-linear, the operator $\mathcal{L}_\psi$ is *tropical linear* (see e.g. [LiSu24] for further development of this tropical functional analysis viewpoint; see also [BLL13]).

Lemma 4.5. *Suppose $\beta > 1$, $\alpha \in (0,1]$, $\phi \in C^{0,\alpha}(I)$, and $n \in \mathbb{N}$. Then*

$$\mathcal{L}_\phi^n(u)(x) - \mathcal{L}_\phi^n(u)(y) \geqslant -K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha \tag{4.4}$$

for all $u \in C^{0,\alpha}(I)$, and all $x, y \in [0,1)$ with $x \leqslant y$.

If, moreover, for all $1 \leqslant i \leqslant n$ the interval $(x,y]$ does not contain $U_\beta^i(1)$, then

$$\left|\mathcal{L}_\phi^n(u)(x) - \mathcal{L}_\phi^n(u)(y)\right| \leqslant K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha. \tag{4.5}$$

Proof. Suppose $u \in C^{0,\alpha}(I)$ and $x, y \in [0,1)$ with $x \leqslant y$. Without loss of generality, we assume $x < y$. By Lemma 4.4 (i), there exists $y' = T_\beta^{-n}(y) \smallsetminus \{1\}$ such that

$$\mathcal{L}_\phi^n(u)(y) = u(y') + S_n\phi(y'). \tag{4.6}$$

By Proposition 3.23 (i), there exists $I^n \in W^n$ containing y'. Since $x < y$, Proposition 3.23 (v) guarantees that $x \in T_\beta^n(I^n)$ as well. Consider $x' \in T_\beta^{-n}(x) \cap I^n$, then $x' \neq 1$, so we have

$$\mathcal{L}_\phi^n(u)(x) \geqslant u(x') + S_n\phi(x'). \tag{4.7}$$

Combining (4.6) and (4.7) gives

$$\mathcal{L}_\phi^n(u)(x) - \mathcal{L}_\phi^n(u)(y) \geqslant S_n\phi(x') + u(x') - S_n\phi(y') - u(y'). \tag{4.8}$$

Since $x', y' \in I^n$, Lemma 3.24 gives

$$S_n\phi(x') - S_n\phi(y') \geqslant -K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha. \tag{4.9}$$

Now $u \in C^{0,\alpha}(I)$, so $u(x') - u(y') \geqslant -|u|_\alpha|x'-y'|^\alpha$ and $|x'-y'| = \beta^{-n}|x-y|$ by Proposition 3.23 (ii), so $u(x') - u(y') \geqslant -|u|_\alpha\beta^{-n\alpha}|x-y|^\alpha$. But $\beta^{-n\alpha} < K_{\alpha,\beta}$, so

$$u(x') - u(y') \geqslant -K_{\alpha,\beta}|u|_\alpha|x-y|^\alpha. \tag{4.10}$$

Combining (4.8), (4.9), and (4.10) gives the required inequality (4.4).

A similar argument can be used to establish the bound (4.5). Specifically, suppose that $x, y \in [0,1)$ and $n \in \mathbb{N}$ are such that $(x,y] \cap \left\{U_\beta(1), \dots, U_\beta^n(1)\right\} = \emptyset$, so that, by Proposition 3.23 (v), if $I^n \in W^n$ then $x \in T_\beta^n(I^n)$ if and only if $y \in T_\beta^n(I^n)$.

By Proposition 3.23 (i) and Lemma 4.4 (i) there exists some $I^n \in W^n$ and $x'' \in I^n$ with $x'' \in T_\beta^{-n}(x) \smallsetminus \{1\}$, such that $\mathcal{L}_\phi^n(u)(x) = S_n\phi(x'') + u(x'')$.

Let $y'' \in T_\beta^{-n}(y) \cap I^n$. An argument analogous to the one above, using Lemma 3.24 and Lemma 4.4 (i), then gives

$$\begin{aligned}\mathcal{L}_\phi^n(u)(x) - \mathcal{L}_\phi^n(u)(y) &\leqslant S_n\phi(x'') + u(x'') - S_n\phi(y'') - u(y'') \leqslant (\beta^\alpha - 1)^{-1}|\phi|_\alpha|x-y|^\alpha + |u|_\alpha|x''-y''|^\alpha \\ &\leqslant (\beta^\alpha - 1)^{-1}|\phi|_\alpha|x-y|^\alpha + |u|_\alpha\beta^{-n\alpha}|x-y|^\alpha \leqslant K_{\alpha,\beta}(|\phi|_\alpha + |u|_\alpha)|x-y|^\alpha,\end{aligned}$$

and (4.5) follows. □

Of particular interest will be the choice $u = \mathbb{0}$, the function that is identically zero on I, and evaluation of (4.4) at the left endpoint of I, which we record as follows:

Corollary 4.6. *Suppose $\beta > 1$ and $\alpha \in (0,1]$. If $\phi \in C^{0,\alpha}(I)$, $n \in \mathbb{N}$, and $y \in [0,1)$, then*

$$\mathcal{L}_\phi^n(\mathbb{0})(0) \geqslant \mathcal{L}_\phi^n(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha.$$

We are now able to find a fixed point u_ϕ of the operator $\mathcal{L}_{\overline{\phi}}$:

Proposition 4.7. *Suppose $\beta > 1$ and $\alpha \in (0,1]$. If $\phi \in C^{0,\alpha}(I)$ then the function $u_\phi \colon [0,1) \to \mathbb{R}$ given by*

$$u_\phi(x) := \limsup_{n \to +\infty} \mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(x), \quad x \in [0,1), \tag{4.11}$$

where $\overline{\phi} := \phi - Q(T_\beta, \phi)$, satisfies the following properties:

(i) *u_ϕ is Borel measurable and $|u_\phi(x)| \leqslant (2 + 3K_{\alpha,\beta})|\phi|_\alpha$ for each $x \in [0,1)$.*

(ii) *If* $a \in (0,1]$ *then* $\lim_{x \nearrow a} u_\phi(x)$ *exists, and provided* $a \neq 1$, *satisfies* $\lim_{x \nearrow a} u_\phi(x) \geqslant u_\phi(a)$. *If* $a \in [0,1)$ *then* $\lim_{x \searrow a} u_\phi(x)$ *exists, and satisfies* $u_\phi(a) \geqslant \lim_{x \searrow a} u_\phi(x)$. *In particular, if* $a \in (0,1)$ *then*

$$\lim_{x \nearrow a} u_\phi(x) \geqslant u_\phi(a) \geqslant \lim_{x \searrow a} u_\phi(x). \tag{4.12}$$

(iii) $|u_\phi(x) - u_\phi(y)| \leqslant K_{\alpha,\beta}|\phi|_\alpha |x-y|^\alpha$ *if* $0 \leqslant x < y < 1$ *satisfy* $(x,y] \cap \mathcal{O}^*_\beta(1) = \emptyset$.

(iv) $\mathcal{L}_{\overline{\phi}}(u_\phi) = u_\phi$.

Proof. For each $n \in \mathbb{N}$ and each $x \in [0,1)$, we write

$$p_n(x) \coloneqq \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(x) \quad \text{and} \quad q_n(x) \coloneqq \sup_{m \geqslant n} p_m(x). \tag{4.13}$$

Note that, for each $x \in [0,1)$, the sequence $\{q_n(x)\}_{n\in\mathbb{N}}$ is non-increasing and

$$u_\phi(x) = \lim_{n\to+\infty} q_n(x) = \limsup_{n\to+\infty} p_n(x).$$

(i) Fix $n \in \mathbb{N}$. By (4.5), p_n is continuous at all points except for $U_\beta(1), \dots, U^n_\beta(1)$, and hence Borel measurable. Combining this with (4.11), u_ϕ is Borel measurable. By Lemma 4.4 (i), there exists $y_n \in T^{-n}_\beta(0) \smallsetminus \{1\}$ such that

$$p_n(0) = \mathcal{L}^n_{\overline{\phi}}(\mathbb{0})(0) = S_n\overline{\phi}(y_n). \tag{4.14}$$

By Proposition 3.23 (i), there exists $I^n = I(a_1, a_2, \dots, a_n)$ containing y_n. By Proposition 3.23 (ii) and (v), y_n is the left-endpoint of I^n, then (3.14) gives

$$y_n = \frac{a_1}{\beta} + \cdots + \frac{a_n}{\beta^n}. \tag{4.15}$$

Define

$$k \coloneqq \min\{i \in \mathbb{N}_0 : a_j = 0 \text{ for all } i+1 \leqslant j \leqslant n\}. \tag{4.16}$$

Case 1. If $k = 0$, we get that $y_n = 0$ and since 0 is a fixed point of T_β,

$$p_n(0) = n\overline{\phi}(0) \leqslant 0. \tag{4.17}$$

Case 2. If $k > 0$, $y_n (\neq 1)$ is the right endpoint of a k-full cylinder $I^k_0 \coloneqq I(a_1, \dots, a_k - 1)$ by Proposition 3.23 (iii) and hence by Proposition 3.23 (iv) there is a T^k_β-fixed point z_n in I^k_0. Thus,

$$S_k\overline{\phi}(z_n) \leqslant kQ\big(T_\beta, \overline{\phi}\big) = 0. \tag{4.18}$$

Since 0 is a fixed point of T_β, $\overline{\phi}(0) \leqslant Q\big(T_\beta, \overline{\phi}\big) = 0$, and hence combining this with the fact that $T^k_\beta(y_n) = 0$ (see (4.15) and (4.16)), we obtain

$$S_n\overline{\phi}(y_n) = S_k\overline{\phi}(y_n) + S_{n-k}\overline{\phi}\big(T^k_\beta(y_n)\big) = S_k\overline{\phi}(y_n) + (n-k)\overline{\phi}(0) \leqslant S_k\overline{\phi}(y_n). \tag{4.19}$$

If $k \geqslant 2$, define the $(k-1)$-cylinder $I^{k-1} \coloneqq I(a_1, a_2, \dots, a_{k-1})$. As $I^n \subseteq I^{k-1}$ and $I^k_0 \subseteq I^{k-1}$ by (3.13), we have y_n, $z_n \in I^{k-1}$, then Lemma 3.24 gives

$$\begin{aligned} S_k\overline{\phi}(y_n) - S_k\overline{\phi}(z_n) &= S_{k-1}\overline{\phi}(y_n) - S_{k-1}\overline{\phi}(z_n) + \overline{\phi}\big(T^{k-1}_\beta(y_n)\big) - \overline{\phi}\big(T^{k-1}_\beta(z_n)\big) \\ &\leqslant (1 + K_{\alpha,\beta})|\phi|_\alpha \big|T^{k-1}_\beta(y_n) - T^{k-1}_\beta(z_n)\big|^\alpha \leqslant (1 + K_{\alpha,\beta})|\phi|_\alpha. \end{aligned} \tag{4.20}$$

Moreover, (4.20) also holds if $k = 1$.

Combining (4.14), (4.19), (4.20), and (4.18) gives

$$p_n(0) = S_n\overline{\phi}(y_n) \leqslant S_k\overline{\phi}(y_n) = S_k\overline{\phi}(y_n) - S_k\overline{\phi}(z_n) + S_k\overline{\phi}(z_n) \leqslant (1 + K_{\alpha,\beta})|\phi|_\alpha. \tag{4.21}$$

Combining Corollary 4.6, (4.17), and (4.21) gives

$$p_n(x) \leqslant p_n(0) + K_{\alpha,\beta}|\phi|_\alpha \leqslant (1 + 2K_{\alpha,\beta})|\phi|_\alpha \quad \text{for all } x \in [0,1), n \in \mathbb{N}, \tag{4.22}$$

so from (4.11) we deduce the upper bound

$$u_\phi(x) \leqslant (1 + 2K_{\alpha,\beta})|\phi|_\alpha \quad \text{for all } x \in [0,1). \tag{4.23}$$

We now seek to derive a lower bound on u_ϕ, via a lower bound on $p_n(x)$. Fix $n \in \mathbb{N}$. First we would like to show there exists $w_n \in [0,1)$ satisfying

$$S_n\overline{\phi}(w_n) \geqslant -|\phi|_\alpha. \tag{4.24}$$

If $|\phi|_\alpha = 0$, in other words ϕ is constant, (4.24) holds for each $w_n \in [0,1)$. Now assume $|\phi|_\alpha > 0$. Note that $Q\bigl(T_\beta, \overline{\phi}\bigr) = 0$, so $Q\bigl(T_\beta, S_n\overline{\phi}\bigr) = 0$. There exists $\mu \in \mathcal{M}(I, T_\beta)$ with $\int \overline{\phi}\,\mathrm{d}\mu > -|\phi|_\alpha$. By Proposition 3.12 (iii), $\mu(\{1\}) = 0$, so $\sup\bigl\{S_n\overline{\phi}(x) : x \in [0,1)\bigr\} > -|\phi|_\alpha$. Consequently, there exists $w_n \in [0,1)$ satisfying (4.24).

Now choose $w_n \in [0,1)$ to satisfy (4.24). Defining $y \coloneqq T_\beta^n(w_n)$, Lemma 4.4 (i) and (4.24) give

$$\mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(y) = \max\bigl\{S_n\overline{\phi}(w) : w = T_\beta^{-n}(y) \smallsetminus \{1\}\bigr\} \geqslant S_n\overline{\phi}(w_n) \geqslant -|\phi|_\alpha. \tag{4.25}$$

Combining (4.13), Corollary 4.6, and (4.25) gives

$$p_n(0) = \mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(0) \geqslant \mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha \geqslant -(1 + K_{\alpha,\beta})|\phi|_\alpha. \tag{4.26}$$

Let y_n and k be as in (4.15) and (4.16). Fix $x \in [0,1)$. When $k = 0$, we get $y_n = 0$ and $p_n(0) = n\overline{\phi}(0)$. Notice that $0, x/\beta^n \in I(0,\dots,0) \in W^n$ and $x/\beta^n \in T_\beta^{-n}(x) \smallsetminus \{1\}$, using Lemma 3.24 and (4.26) gives

$$p_n(x) \geqslant S_n\overline{\phi}(x/\beta^n) = S_n\overline{\phi}(x/\beta^n) - S_n\overline{\phi}(0) + p_n(0) \geqslant -(1 + 2K_{\alpha,\beta})|\phi|_\alpha. \tag{4.27}$$

When $k > 0$, by Proposition 3.23 (iii) and (4.15), we have the full n-cylinder

$$J^n \coloneqq I(0,\dots,0,a_1,a_2,\dots,a_k - 1) \in W_0^n,$$

with the right endpoint equal to y_n/β^{n-k}. Since J^n is full, then there exists $z_n \in T_\beta^{-n}(x) \cap J^n$. Noting that $z_n \neq 1$, then by (4.13) and Lemma 4.4 (i),

$$p_n(x) = \mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(x) = \max\bigl\{S_n\overline{\phi}(z) : z \in T_\beta^{-n}(x) \smallsetminus \{1\}\bigr\} \geqslant S_n\overline{\phi}(z_n). \tag{4.28}$$

If $n \geqslant 2$, denote $J^{n-1} \coloneqq I(0,\dots,0,a_1,a_2,\dots,a_{k-1}) \in W^{n-1}$, in particular, $J^{n-1} = I(0,\dots,0) \in W^{n-1}$ if $k = 1$. By (4.15), we have $y_n/\beta^{n-k} \in J^{n-1}$, and since $J^n \subseteq J^{n-1}$, we have $z_n \in J^{n-1}$. Applying Lemma 3.24, we have

$$\begin{aligned} S_n\overline{\phi}(z_n) - S_n\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) &= S_{n-1}\overline{\phi}(z_n) - S_{n-1}\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) + \overline{\phi}\bigl(T_\beta^{n-1}(z_n)\bigr) - \overline{\phi}\bigl(T_\beta^{n-1}\bigl(y_n/\beta^{n-k}\bigr)\bigr) \\ &\geqslant -(1 + K_{\alpha,\beta})|\phi|_\alpha\bigl|T_\beta^{n-1}(z_n) - T_\beta^{n-1}\bigl(y_n/\beta^{n-k}\bigr)\bigr|^\alpha \geqslant -(1 + K_{\alpha,\beta})|\phi|_\alpha. \end{aligned} \tag{4.29}$$

Moreover, (4.29) also holds if $n = 1$.

Now by (3.1),

$$S_n\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) = S_k\overline{\phi}(y_n) + S_{n-k}\overline{\phi}\bigl(y_n/\beta^{n-k}\bigr) = S_k\overline{\phi}(y_n) + \sum_{i=1}^{n-k}\overline{\phi}\bigl(y_n/\beta^i\bigr). \tag{4.30}$$

Note that $\overline{\phi}\bigl(y_n/\beta^i\bigr) - \overline{\phi}(0) \geqslant -|\phi|_\alpha\bigl(y_n/\beta^i\bigr)^\alpha$ for $1 \leqslant i \leqslant n-k$, so

$$\sum_{i=1}^{n-k}\overline{\phi}\bigl(y_n/\beta^i\bigr) \geqslant (n-k)\overline{\phi}(0) - |\phi|_\alpha\sum_{i=1}^{n-k}\bigl(y_n/\beta^i\bigr)^\alpha \geqslant (n-k)\overline{\phi}(0) - K_{\alpha,\beta}|\phi|_\alpha. \tag{4.31}$$

Combining (4.28), (4.29), (4.30), and (4.31) gives

$$p_n(x) \geqslant (n-k)\overline{\phi}(0) - (1 + 2K_{\alpha,\beta})|\phi|_\alpha + S_k\overline{\phi}(y_n). \tag{4.32}$$

However, (4.14) and (4.19) together give

$$(n-k)\overline{\phi}(0) + S_k\overline{\phi}(y_n) = p_n(0). \tag{4.33}$$

So combining (4.32), (4.33), and (4.26) gives

$$p_n(x) \geqslant p_n(0) - (1 + 2K_{\alpha,\beta})|\phi|_\alpha \geqslant -(2 + 3K_{\alpha,\beta})|\phi|_\alpha. \tag{4.34}$$

Therefore, (4.27), (4.34), (4.11), and (4.14) give

$$u_\phi(x) \geqslant -(2+3K_{\alpha,\beta})|\phi|_\alpha. \tag{4.35}$$

The bounds (4.23) and (4.35) together give the required inequality $|u_\phi(x)| \leqslant (2+3K_{\alpha,\beta})|\phi|_\alpha$, so (i) is proved.

(ii) If $0 \leqslant x < y < 1$ then taking $u = \mathbb{0}$ in Lemma 4.5 and using Lemma 4.4 (i) give

$$\mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(x) \geqslant \mathcal{L}_{\overline{\phi}}^n(\mathbb{0})(y) - K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha,$$

and taking the limit supremum, together with (4.11), gives

$$u_\phi(x) \geqslant u_\phi(y) - K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha. \tag{4.36}$$

If $a \in [0,1)$ then in particular (4.36) holds for all $a < x < y < 1$, so taking $\liminf_{x\searrow a}$ gives

$$\liminf_{x\searrow a} u_\phi(x) \geqslant u_\phi(y) - K_{\alpha,\beta}|\phi|_\alpha|a-y|^\alpha, \tag{4.37}$$

and taking $\limsup_{y\searrow a}$ in (4.37) gives

$$\liminf_{x\searrow a} u_\phi(x) \geqslant \limsup_{y\searrow a} u_\phi(y), \tag{4.38}$$

so $\lim_{x\searrow a} u_\phi(x)$ exists, as required. Now setting $x = a$ in (4.36), and taking $\lim_{y\searrow a} u_\phi(y)$, gives that

$$u_\phi(a) \geqslant \lim_{y\searrow a} u_\phi(y), \tag{4.39}$$

as required.

If $a \in (0,1]$, an analogous argument shows that $\lim_{x\nearrow a} u_\phi(x)$ exists, and if moreover $a \neq 1$ then

$$\lim_{x\nearrow a} u_\phi(x) \geqslant u_\phi(a). \tag{4.40}$$

If $a \in (0,1)$, the required inequality (4.12) is immediate from (4.39) and (4.40).

(iii) Now suppose $0 \leqslant x < y < 1$ with $(x,y] \cap \mathcal{O}_\beta^*(1) = \emptyset$. For each $\varepsilon > 0$, by (4.13) and (4.11) there exists $N \in \mathbb{N}$ such that $|p_N(x) - u_\phi(x)| < \varepsilon$ and $q_N(y) - u_\phi(y) < \varepsilon$, so

$$u_\phi(x) - u_\phi(y) < p_N(x) - q_N(y) + 2\varepsilon \leqslant p_N(x) - p_N(y) + 2\varepsilon \leqslant K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha + 2\varepsilon, \tag{4.41}$$

where the final inequality uses (4.5). Similarly, there exists $M \in \mathbb{N}$ such that $|p_M(y) - u_\phi(y)| < \varepsilon$ and $q_M(x) - u_\phi(x) < \varepsilon$, and an analogous calculation gives

$$u_\phi(x) - u_\phi(y) \geqslant -K_{\alpha,\beta}|\phi|_\alpha|x-y|^\alpha - 2\varepsilon. \tag{4.42}$$

Since $\varepsilon > 0$ was arbitrary, (iii) follows from (4.41) and (4.42).

(iv) If $x \in [0,1)$, then by Lemma 4.4 (ii), (iii), (4.13), and the boundedness of $p_n(x)$ and $q_n(x)$ (cf. (4.22) and (4.34)),

$$\begin{aligned}\mathcal{L}_{\overline{\phi}}(u_\phi)(x) &= \mathcal{L}_{\overline{\phi}}\Big(\lim_{n\to+\infty} q_n\Big)(x) = \lim_{n\to+\infty} \mathcal{L}_{\overline{\phi}}\Big(\sup_{m\geqslant n} \mathcal{L}_{\overline{\phi}}^m(\mathbb{0})\Big)(x) \\ &= \lim_{n\to+\infty} \sup_{m\geqslant n} \mathcal{L}_{\overline{\phi}}^{m+1}(\mathbb{0})(x) = \lim_{n\to+\infty} q_{n+1}(x) = u_\phi(x).\end{aligned}$$

□

The following construction of the regularisations of the function u_ϕ is key to our revelation theorem (Theorem 4.12).

Definition 4.8. Fix $\beta > 1$ and $\alpha \in (0,1]$. For each $\phi \in C^{0,\alpha}(I)$, let u_ϕ be the function defined in (4.11). Since U_β is left-continuous and upper semi-continuous on $(0,1]$, we define a *sub-action* for (U_β, ϕ) by

$$u_{\beta,\phi}^-(x) := \begin{cases} u_\phi(0) & \text{if } x = 0, \\ \lim_{y\nearrow x} u_\phi(y) & \text{if } x \in (0,1]. \end{cases} \tag{4.43}$$

Since T_β is right-continuous and lower semi-continuous on $[0,1)$, we define a sub-action for (T_β, ϕ) by

$$u^+_{\beta,\phi}(x) := \begin{cases} \lim_{y\searrow x} u_\phi(y) & \text{if } x \in [0,1), \\ \lim_{y\searrow T_\beta(1)} u_\phi(y) - \overline{\phi}(1) & \text{if } x = 1. \end{cases} \tag{4.44}$$

We define the *left-continuous revealed version* $\widetilde{\phi}^-$ and the *right-continuous revealed version* $\widetilde{\phi}^+$ by

$$\widetilde{\phi}^- := \overline{\phi} + u^-_{\beta,\phi} - u^-_{\beta,\phi} \circ U_\beta, \tag{4.45}$$

$$\widetilde{\phi}^+ := \overline{\phi} + u^+_{\beta,\phi} - u^+_{\beta,\phi} \circ T_\beta. \tag{4.46}$$

We are now able to prove a Mañé lemma[26] for beta-transformations, involving the above sub-actions and revealed versions.

Theorem 4.9 (Mañé lemma for beta-transformations). *If $\beta > 1$, $\alpha \in (0,1]$, and $\phi \in C^{0,\alpha}(I)$, then:*

(i) *$u^-_{\beta,\phi}$ is bounded, left-continuous, and upper semi-continuous, on $(0,1]$, and $u^+_{\beta,\phi}$ is bounded, right-continuous, and lower semi-continuous, on $[0,1)$.*

(ii) *$\widetilde{\phi}^- \leqslant 0$ and $\widetilde{\phi}^+ \leqslant 0$ on I. The function $\widetilde{\phi}^-$ is left-continuous on $(0,1]$, and $\widetilde{\phi}^+$ is right-continuous on $[0,1)$.*

(iii) *If the closed interval $[x,y] \subseteq I$ is disjoint from the orbit $\mathcal{O}^*_\beta(1)$, then*

$$\bigl|u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(y)\bigr| \leqslant K_{\alpha,\beta}|\phi|_\alpha |x-y|^\alpha \quad \textit{and} \tag{4.47}$$

$$\bigl|u^+_{\beta,\phi}(x) - u^+_{\beta,\phi}(y)\bigr| \leqslant K_{\alpha,\beta}|\phi|_\alpha |x-y|^\alpha. \tag{4.48}$$

Proof. (i) follows immediately from (4.43), (4.44), and Proposition 4.7 (i) and (ii).

(ii) By (4.1) and Proposition 4.7 (iv),

$$u_\phi(x) = \max\bigl\{\overline{\phi}(y) + u_\phi(y) : y \in T_\beta^{-1}(x) \smallsetminus \{1\}\bigr\} \quad \text{for } x \in [0,1). \tag{4.49}$$

So for all $x \in [0,1)$, we have

$$\widetilde{\phi}(x) := \overline{\phi}(x) + u_\phi(x) - u_\phi(T_\beta(x)) \leqslant 0. \tag{4.50}$$

Combining this inequality with (4.46), (4.44), and Lemma 3.4 (i), for all $x \in [0,1)$, we obtain

$$\widetilde{\phi}^+(x) = \overline{\phi}(x) + u^+_{\beta,\phi}(x) - u^+_{\beta,\phi}(T_\beta(x)) = \lim_{y\searrow x} \widetilde{\phi}(y) \leqslant 0. \tag{4.51}$$

By (4.46) and (4.44),

$$\widetilde{\phi}^+(1) = \overline{\phi}(1) + u^+_{\beta,\phi}(1) - u^+_{\beta,\phi}(T_\beta(1)) = 0. \tag{4.52}$$

Combining (4.51) and (4.52) gives $\widetilde{\phi}^+ \leqslant 0$ on I and the right-continuity of $\widetilde{\phi}^+$ on $[0,1)$. Combining (4.50), (4.45), (4.43), and Lemma 3.4 (iv), for all $x \in (0,1]$, we obtain

$$\widetilde{\phi}^-(x) = \overline{\phi}(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(U_\beta(x)) = \lim_{y\nearrow x} \widetilde{\phi}(y) \leqslant 0. \tag{4.53}$$

Since $U_\beta(0) = 0$ (see (3.2)), by (4.45),

$$\widetilde{\phi}^-(0) = \overline{\phi}(0) \leqslant 0. \tag{4.54}$$

Combining (4.54) and (4.53) gives $\widetilde{\phi}^- \leqslant 0$ on I and the left-continuity of $\widetilde{\phi}^-$ on $(0,1]$.

(iii) This follows immediately from (4.43), (4.44), and Proposition 4.7 (iii). □

[26]Note that the statement of the Mañé cohomology lemma for beta-transformations (Theorem 4.9) takes a rather different form from the analogous result for open expanding maps (Theorem 2.6): in Theorem 4.9 the continuity properties are emphasised, as these are particularly delicate, whereas the cohomology properties are implicit in the definitions (4.45), and (4.46).

Remark 4.10. If β is a beta-number, then $\mathcal{O}_\beta^*(1)$ is a finite set, and by Theorem 4.9 (iii), the functions $u_{\beta,\phi}^-$ and $u_{\beta,\phi}^+$ are piecewise α-Hölder.

Remark 4.11. The approach to proving the Mañé lemma for beta-transformations (Theorem 4.9) was quite different from that used to prove the analogous result for open distance-expanding maps (Theorem 2.6). Specifically, the transitivity of beta-transformations means that we are here able to construct a *calibrated* sub-action, something that is in general not possible for non-transitive systems. On the other hand, due to the lack of openness of the map, the function u_ϕ defined in (4.11) can only be shown to be Hölder continuous away from the critical orbit, rather than on the whole of I; however a special property of beta-transformations (namely that if $x < y$ then any inverse branch for y is also an inverse branch for x), led to the inequality (4.4), and ultimately to the proof that u_ϕ has both one-sided limits existing everywhere. Note that the lack of openness also means that there is no shadowing lemma analogue of Lemma 2.4, therefore to prove that $-\infty < u_\phi < +\infty$ we were unable to pattern the approach on that used to prove Theorem 2.6, but instead employed a more technical argument depending on various special characteristics of beta-transformations.

The importance of Theorem 4.9 is in allowing us to establish a form of *revelation theorem* (cf. [Je19, Section 5]): the following Theorem 4.12 localises the support of a maximizing measure as lying in the union of the zero sets of the revealed versions $\widetilde{\phi}^-$ and $\widetilde{\phi}^+$, and reveals that individual points in the support of such a measure have their full orbit, under either U_β or T_β, contained in either $\big(\widetilde{\phi}^+\big)^{-1}(0)$ or $\big(\widetilde{\phi}^-\big)^{-1}(0)$ respectively. This latter fact will be exploited in our proof of Theorem H, more specifically in establishing the key Lemma 5.23.

Theorem 4.12 (Revelation theorem). *If $\beta > 1$, $\alpha \in (0,1]$, $\phi \in C^{0,\alpha}(I)$, and $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$, then:*

(i) *If $x \in \operatorname{supp}\mu$, then either $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}_\beta^*(x)$, or $\widetilde{\phi}^+ \equiv 0$ on $\mathcal{O}_\beta(x)$.*

(ii) *If $1 \in \operatorname{supp}\mu$ then $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}_\beta^*(1)$.*

In particular, $\operatorname{supp}\mu \subseteq \big(\widetilde{\phi}^+\big)^{-1}(0) \cup \big(\widetilde{\phi}^-\big)^{-1}(0)$.

Proof. (i) Since $\widetilde{\phi}^- = \overline{\phi} + u_{\beta,\phi}^- - u_{\beta,\phi}^- \circ U_\beta$ (see (4.45)) where $u_{\beta,\phi}^-$ is bounded and Borel measurable (see Proposition 4.7 (i) and (4.43)), and $\overline{\phi} = \phi - Q(T_\beta, \phi) = \phi - Q(U_\beta, \phi)$ (cf. Proposition 3.28 (ii)) is normalised, it follows from Lemma 4.2 (i) and (ii) that $Q\big(U_\beta, \widetilde{\phi}^-\big) = Q\big(U_\beta, \overline{\phi}\big) = 0$ and $\mathcal{M}_{\max}\big(U_\beta, \phi\big) = \mathcal{M}_{\max}\big(U_\beta, \overline{\phi}\big) = \mathcal{M}_{\max}(U_\beta, \widetilde{\phi}^-)$. So

$$\int \widetilde{\phi}^- \,\mathrm{d}\mu = 0. \tag{4.55}$$

Suppose $x \in \operatorname{supp}\mu$, so that $\mu((x-\varepsilon, x+\varepsilon)) > 0$ for all $\varepsilon > 0$. Thus, at least one of the following two cases will occur: either $\mu((x-\varepsilon, x]) > 0$ for all $\varepsilon > 0$, or $\mu((x, x+\varepsilon)) > 0$ for all $\varepsilon > 0$.

Firstly, let us suppose that

$$\mu((x-\varepsilon, x]) > 0 \text{ for all } \varepsilon > 0, \tag{4.56}$$

and in this case we aim to show that

$$\widetilde{\phi}^- \equiv 0 \text{ on } \mathcal{O}_\beta^*(x). \tag{4.57}$$

If $x = 0$, then (4.56) implies that $\mu(\{0\}) > 0$, and since $\widetilde{\phi}^- \leqslant 0$, it follows that

$$0 = \int \widetilde{\phi}^- \,\mathrm{d}\mu \leqslant \mu(\{0\})\widetilde{\phi}^-(0),$$

so that $\widetilde{\phi}^-(0) = 0$, in other words, $\widetilde{\phi}^- \equiv 0$ on $\{0\} = \mathcal{O}_\beta^*(0)$, so (4.57) holds.

If $x \neq 0$, we first claim that

$$\mu((y-\varepsilon, y]) > 0 \text{ for all } y \in \mathcal{O}_\beta^*(x) \text{ and } \varepsilon > 0. \tag{4.58}$$

To prove (4.58), note that $0 \notin \mathcal{O}^*_\beta(x)$ since $x \neq 0$ (see Proposition 3.12 (i)). Assume that $y \in \mathcal{O}^*_\beta(x)$, so that $y = U^k_\beta(x)$ for some $k \in \mathbb{N}$. By Lemma 3.4 (i), for all $\varepsilon > 0$ there exists $\delta > 0$ such that $U^k_\beta((x-\delta, x]) \subseteq (y-\varepsilon, y]$. But $\mu \in \mathcal{M}(I, U_\beta)$, so (4.56) implies that $\mu((y-\varepsilon, y]) = \mu\big(U_\beta^{-k}(y-\varepsilon, y]\big) \geqslant \mu((x-\delta, x]) > 0$. So (4.58) holds.

Now $\widetilde{\phi}^-$ is left-continuous by Theorem 4.9 (ii), so if $y \in \mathcal{O}^*_\beta(x)$ were such that $\widetilde{\phi}^-(y) < 0$, then there would exist $\rho, \varepsilon > 0$ with $\widetilde{\phi}^-|_{(y-\varepsilon,y]} < -\rho$, and (4.58) would imply that

$$\int \widetilde{\phi}^- \,\mathrm{d}\mu \leqslant \int_{(y-\varepsilon,y]} \widetilde{\phi}^- \,\mathrm{d}\mu < -\rho \cdot \mu((y-\varepsilon, y]) < 0,$$

which contradicts (4.55). So $\widetilde{\phi}^-(y)$ cannot be strictly negative, but on the other hand $\widetilde{\phi}^- \leqslant 0$ by Theorem 4.9 (ii), thus $\widetilde{\phi}^-(y) = 0$. Since y was an arbitrary point in $\mathcal{O}^*_\beta(x)$, (4.57) holds.

Secondly, let us suppose that

$$\mu((x, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0, \tag{4.59}$$

and in this case we aim to show that $\widetilde{\phi}^+ \equiv 0$ on $\mathcal{O}_\beta(x)$. Note that (4.59) gives $x \neq 1$, and consequently $1 \notin \mathcal{O}_\beta(x)$ by Proposition 3.12 (i). By arguments analogous to the ones above, Lemma 3.4 (i) first implies that $\mu([y, y+\varepsilon)) > 0$ for all $y \in \mathcal{O}_\beta(x)$ and $\varepsilon > 0$, and then the right-continuity of $\widetilde{\phi}^+$ (see Theorem 4.9 (ii)) implies that $\widetilde{\phi}^+(y) = 0$ for each $y \in \mathcal{O}_\beta(x)$, as required.

(ii) If $1 \in \operatorname{supp} \mu$, then (4.56) holds for $x = 1$, and the argument in (i) above gives (4.57), in other words, $\widetilde{\phi}^- \equiv 0$ on $\mathcal{O}^*_\beta(1)$, as required. □

Remark 4.13. A consequence of Theorem 4.9, and a counterpoint to Theorem 4.12, is that ϕ-maximizing measures can be characterised in terms of the support of their pushforward under π^*_β: specifically, for all $\beta > 1$, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$, there exists a closed subset $K \subseteq X_\beta$ such that a U_β-invariant measure μ belongs to $\mathcal{M}_{\max}(U_\beta, \phi)$ if and only if $\operatorname{supp} G_\beta(\mu) \subseteq K$. To see this, recall from Proposition 3.27 that $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ if and only if $G_\beta(\mu) \in \mathcal{M}_{\max}\big(\sigma|_{X_\beta}, \phi \circ h_\beta\big)$, and then the continuity of $\sigma|_{X_\beta}$ and $\phi \circ h_\beta$ means that the existence of such a K (a so-called *maximizing set* in the terminology of Morris [Mo07]) is guaranteed (using [Mo07, Theorem 1, Proposition 1]) if

$$\sup_{n\in\mathbb{N}} \sup_{A\in X_\beta} S^\sigma_n \big(\overline{\phi} \circ h_\beta\big)(A) < +\infty. \tag{4.60}$$

The bound (4.60) can be proved using $\widetilde{\phi}^-$ and $\widetilde{\phi}^+$, since the sub-actions $u^-_{\beta,\phi}$, $u^+_{\beta,\phi}$ are bounded, together with the fact that $X_\beta = \pi_\beta(I) \cup \pi^*_\beta(I)$ implied by (3.15) and Lemma 3.26 (ii). More precisely, if $A \in \pi^*_\beta(I)$, with $A = \pi^*_\beta(x)$, then parts (iv) and (v) of Proposition 3.9, together with (4.45), give

$$S^\sigma_n \big(\overline{\phi} \circ h_\beta\big)(A) = S^{U_\beta}_n \overline{\phi}(x) = S^{U_\beta}_n \widetilde{\phi}^-(x) + u^-_\phi\big(U^n_\beta(x)\big) - u^-_\phi(x) \quad \text{ for all } n \in \mathbb{N},$$

and using that $\widetilde{\phi}^- \leqslant 0$ (by Theorem 4.9), together with the bound on the modulus of u_ϕ from Proposition 4.7 (i) and the definition of u^-_ϕ, yields

$$S^\sigma_n \big(\overline{\phi} \circ h_\beta\big)(A) \leqslant 2(3K_{\alpha,\beta} + 2)|\phi|_\alpha \quad \text{ for all } n \in \mathbb{N}. \tag{4.61}$$

If $A \in \pi_\beta(I)$ then a similar argument, using $\widetilde{\phi}^+$, gives

$$S^\sigma_n \big(\overline{\phi} \circ h_\beta\big)(A) \leqslant 2(3K_{\alpha,\beta} + 3)|\phi|_\alpha \quad \text{ for all } n \in \mathbb{N} \tag{4.62}$$

(the right-hand side of (4.62) differing from that of (4.61) due to the way u^+_ϕ is defined at the point 1), and (4.61), (4.62) together imply (4.60).

Remark 4.14. In view of Remark 4.13, that characterises maximizing measures in terms of some support being contained in a maximizing set, a natural question is whether the condition that $\operatorname{supp} \mu \subseteq \big(\widetilde{\phi}^+\big)^{-1}(0) \cup \big(\widetilde{\phi}^-\big)^{-1}(0)$, which by Theorem 4.12 is necessary for a U_β-invariant measure to be ϕ-maximizing, is actually *sufficient*. The answer is no, as we explain below; note that this contrasts with the situation for open distance-expanding maps, where the Mañé lemma (Theorem 2.6) guarantees that the sub-action (and hence the revealed version) is continuous.

Let $\beta \approx 2.48119$ be the largest root of the cubic polynomial $\zeta^3 - 2\zeta^2 - 2\zeta + 2$. This is a non-simple beta-number, with $\pi_\beta(1) = 2(10)^\infty$.

Define the fixed point $z \coloneqq h_\beta((1)^\infty) = \frac{1}{\beta-1} \approx 0.675$, and the two period-2 points $x \coloneqq h_\beta((10)^\infty) = \frac{\beta}{\beta^2-1} \approx 0.481$, $y \coloneqq h_\beta((01)^\infty) = \frac{1}{\beta^2-1} \approx 0.194$. We will exhibit Lipschitz functions ϕ such that the periodic measure $\mu \coloneqq \frac{1}{2}(\delta_x + \delta_y)$ is not (U_β, ϕ)-maximizing, yet its support $\{x, y\}$ is contained in $\big(\widetilde{\phi}^+\big)^{-1}(0) \cup \big(\widetilde{\phi}^-\big)^{-1}(0)$.

Defining $\tau(s) \coloneqq (s+1)/\beta$, let $x_1 \coloneqq h_\beta(1(10)^\infty) = \tau(x) = \frac{\beta^2+\beta-1}{\beta^3-\beta} \approx 0.597$, and define sequences

$$t_i \coloneqq h_\beta\big((1)^i 2(10)^\infty\big) = h_\beta\big((1)^i \pi_\beta(1)\big) = \tau^i(1) = \beta^{-i}\left(2 + \sum_{j=1}^{i-1} \beta^j\right), \quad i \in \mathbb{N},$$

$$y_i \coloneqq h_\beta\big((1)^{i-1} 0(01)^\infty\big) = \tau^{i-1}(y/\beta) = \frac{\beta^{i+1} + \beta^i - \beta^2 - \beta + 1}{\beta^i(\beta^2 - 1)}, \quad i \in \mathbb{N}.$$

Note in particular that $\{t_i\}_{i\in\mathbb{N}}$ is a backwards orbit of 1 (under U_β, but not T_β) converging to z, that $\{y_i\}_{i\in\mathbb{N}}$ is a backwards orbit of y (under both T_β and U_β), also converging to z, that $T_\beta^{-1}(x) = U_\beta^{-1}(x) = \{y, x_1, 1\}$, and that the relative ordering of the various points is

$$0 < y_1 < y < y_2 < x < y_3 < x_1 < y_4 < y_5 < y_6 < \cdots < z < \cdots < t_3 < t_2 < t_1 < 1.$$

For $\alpha \in (0,1]$, let $\phi \in C^{0,\alpha}(I)$ be non-positive, with $\phi(y) = -1$ and $\phi(x_1) = -2$, and identically zero on the points x, 1, and all points in the two sequences $\{t_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$. (For concreteness, ϕ might be chosen to be the function that is identically zero on the intervals $[0, y_1]$, $[y_2, y_3]$, and $[y_4, 1]$, and affine on each of $[y_1, y]$, $[y, y_2]$, $[y_3, x_1]$, $[x_1, y_4]$, with $\phi(y) = -1$ and $\phi(x_1) = -2$, though our analysis does not assume this.) Note that ϕ is normalised, i.e., $Q(U_\beta, \phi) = 0$, since it attains its maximum value 0 at the fixed point z. On the other hand, $\int \phi \,\mathrm{d}\mu = -1/2 < Q(U_\beta, \phi)$, so $\mu \notin \mathcal{M}_{\max}(U_\beta, \phi)$.

For this value of β (and indeed whenever β is a beta-number), it can be shown that the sub-actions $u^-_{\beta,\phi}$, $u^+_{\beta,\phi}$, defined in (4.43), (4.44), satisfy $u^-_{\beta,\phi}(r) = \limsup_{n\to+\infty} \max\big\{S_n^{U_\beta}\phi(s) : s \in U_\beta^{-n}(r)\big\}$ for all $r \in (0,1]$, and $u^+_{\beta,\phi}(r) = \limsup_{n\to+\infty} \max\big\{S_n^{T_\beta}\phi(s) : s \in T_\beta^{-n}(r)\big\}$ for all $r \in [0,1)$. Now for each $n \in \mathbb{N}$, the point $y_n \in U_\beta^{-n}(y) \cap T_\beta^{-n}(y)$, and $S_n^{U_\beta}\phi(y_n) = 0 = S_n^{T_\beta}\phi(y_n)$, thus $u^-_{\beta,\phi}(y) = 0 = u^+_{\beta,\phi}(y)$. For each $n \geqslant 2$, the point $t_{n-1} \in U_\beta^{-n}(x)$, and $S_n^{U_\beta}\phi(t_{n-1}) = 0$, thus $u^+_{\beta,\phi}(x) = 0$. Since the point 1 does not have any pre-images under T_β, the points t_{n-1} do not belong to $T_\beta^{-n}(x)$. Apart from 1, the other two pre-images of x are y and x_1, and the inequality $\phi(y) > \phi(x_1)$, together with the fact that $\phi \equiv 0$ on the backwards orbit $\{y_n\}_{n\in\mathbb{N}}$ of y, means that $\max\big\{S_n^{T_\beta}\phi(s) : s \in T_\beta^{-n}(x)\big\} = \phi(y) = -1$ for all $n \in \mathbb{N}$, and hence $u^+_{\beta,\phi}(x) = -1$.

Finally, we can evaluate the left-continuous and right-continuous revealed versions at, respectively, the points x and y, to find (cf. (4.45) and (4.46)) that

$$\widetilde{\phi}^-(x) = \overline{\phi}(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(U_\beta(x)) = \phi(x) + u^-_{\beta,\phi}(x) - u^-_{\beta,\phi}(y) = 0 + 0 + 0 = 0 \quad \text{and}$$

$$\widetilde{\phi}^+(y) = \overline{\phi}(y) + u^+_{\beta,\phi}(y) - u^+_{\beta,\phi}(T_\beta(y)) = \phi(y) + u^+_{\beta,\phi}(y) - u^+_{\beta,\phi}(x) = -1 + 0 + 1 = 0,$$

so $\operatorname{supp}\mu = \{x, y\} \subseteq \big(\widetilde{\phi}^+\big)^{-1}(0) \cup \big(\widetilde{\phi}^-\big)^{-1}(0)$, as claimed.

5. Individual TPO: beta-transformations

The primary purpose of this section is to prove, in the context of beta-transformations, the individual typical periodic optimization theorems stated in Section 1 (more precisely, Theorems D, E, G, and H). Our proofs will exploit the properties of two fundamental subsets of $C^{0,\alpha}(I)$, the *critical* *set* $\mathrm{Crit}^\alpha(\beta)$, and the *regular set* $\mathcal{R}^\alpha(\beta)$. In Subsection 5.1 we introduce the notion of an *emergent* parameter β, establish several alternative characterisations, and show that the set of emergent parameters constitutes a small subset of $(1,+\infty)$. Some characterisations of $\mathrm{Crit}^\alpha(\beta)$, consisting of those functions for which the critical orbit is maximizing, are given in Subsection 5.2. In Subsection 5.3, we first define the regular set $\mathcal{R}^\alpha(\beta) \subseteq C^{0,\alpha}(I)$, which consists of those $\phi \in C^{0,\alpha}(I)$ satisfying $\phi|_{H^\gamma_\beta} \in \mathrm{Lock}^\alpha\bigl(T_\beta|_{H^\gamma_\beta}\bigr)$ for all simple beta-numbers $\gamma \in (1,\beta)$, and then show that it is a dense subset of $C^{0,\alpha}(I)$ (Theorem 5.16). In Subsection 5.4 we focus on emergent parameters β; in the absence of an Individual TPO theorem, we establish a structural theorem (Theorem 5.22) identifying the critical set $\mathrm{Crit}^\alpha(\beta)$ as the source of any possible failure of typical periodic optimization. Lastly, in Subsection 5.5, we prove the various Individual TPO theorems (Theorems D, E, G, and H).

5.1. **Emergent parameters.**

Definition 5.1. A parameter $\beta > 1$ will be called *emergent* if

$$\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))} \cap \mathcal{S}_\gamma = \emptyset \quad \text{ for all } \gamma \in (1,\beta).$$

The set of emergent parameters will be denoted by E, and called the *emergent set*.

Remark 5.2. The intuition behind the terminology *emergent*, as described in Section 1, and in view of Lemma 3.16, is that the particular symbolic dynamics of the upper beta-expansion $\pi^*_\beta(1)$ has newly *emerged* at the value β, in the sense that it does not resemble any that has been witnessed for $\gamma \in (1,\beta)$.

Recalling from (3.15) that $Z_\beta = \bigl\{x \in I : \pi_\beta(x) \neq \pi^*_\beta(x)\bigr\}$, the following proposition gives two alternative characterisations of emergent parameters.

Proposition 5.3. *For $\beta > 1$, the following are equivalent:*

(i) *β is emergent.*
(ii) *$\overline{\mathcal{O}^*_\beta(1)} \cap H^\gamma_\beta \subseteq Z_\beta$ for each $\gamma \in (1,\beta)$.*
(iii) *$\bigl(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}, \sigma\bigr)$ is minimal.*

Proof. (i) implies (ii): Assume that β is emergent. Let us suppose, for a contradiction, that the result is false, i.e., that there exists $\gamma \in (1,\beta)$ and $x \in I$ satisfying

$$x \in \Bigl(\overline{\mathcal{O}^*_\beta(1)} \cap H^\gamma_\beta\Bigr) \smallsetminus Z_\beta.$$

By Proposition 3.9 (iii), we have $\mathcal{O}^\sigma\bigl(\pi^*_\beta(1)\bigr) = \pi^*_\beta\bigl(\mathcal{O}^*_\beta(1)\bigr)$. Since the map π^*_β is continuous at x (see Lemma 3.26 (iv)), we have $\pi^*_\beta(x) \in \overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}$. Since $\pi_\beta|_{H^\gamma_\beta}$ is a homeomorphism (see Lemma 3.20) and $x \in H^\gamma_\beta$, we have $\pi_\beta(x) \in \mathcal{S}_\gamma$. Hence, since $x \notin Z_\beta$ we have $\pi_\beta(x) = \pi^*_\beta(x) \in \overline{\mathcal{O}^\sigma(\pi^*_\beta(1))} \cap \mathcal{S}_\gamma$, which contradicts (see Definition 5.1) the fact that β is emergent.

(ii) implies (i): Assume that β is non-emergent. Then there exists $\gamma \in (1,\beta)$ such that $\mathcal{K}_\gamma \coloneqq \mathcal{S}_\gamma \cap \overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}$ is a non-empty closed subset of $\mathcal{S}_\gamma$ with $\sigma(\mathcal{K}_\gamma) \subseteq \mathcal{K}_\gamma$, by Definition 5.1. Then by Lemma 3.20 and Proposition 3.9 (v), $h_\beta(\mathcal{K}_\gamma)$ is a non-empty closed subset of H^γ_β with $T_\beta(h_\beta(\mathcal{K}_\gamma)) \subseteq h_\beta(\mathcal{K}_\gamma)$. Since h_β is continuous and $\mathcal{O}^*_\beta(1) = h_\beta\bigl(\mathcal{O}^\sigma\bigl(\pi^*_\beta(1)\bigr)\bigr)$ (see Proposition 3.9 (x), (iii), and (iv)), we obtain $h_\beta(\mathcal{K}_\gamma) \subseteq \overline{\mathcal{O}^*_\beta(1)}$. Hence $h_\beta(\mathcal{K}_\gamma) \subseteq \overline{\mathcal{O}^*_\beta(1)} \cap H^\gamma_\beta$.

If $h_\beta(\mathcal{K}_\gamma) \subseteq Z_\beta$, then from the fact that $T_\beta(h_\beta(\mathcal{K}_\gamma)) \subseteq h_\beta(\mathcal{K}_\gamma)$, and Lemma 3.26 (i), we have both that $0 \in h_\beta(\mathcal{K}_\gamma)$ and $0 \notin Z_\beta$, which is a contradiction. If on the other hand $h_\beta(\mathcal{K}_\gamma) \not\subseteq Z_\beta$ then there exists $x \in h_\beta(\mathcal{K}_\gamma) \smallsetminus Z_\beta$. In both cases, $\Bigl(\overline{\mathcal{O}^*_\beta(1)} \cap H^\gamma_\beta\Bigr) \smallsetminus Z_\beta$ is non-empty.

(i) implies (iii): Assume that β is emergent. Fix an arbitrary $A \in \overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}$. If $\pi^*_\beta(1) \in \overline{\mathcal{O}^\sigma(A)}$, then $\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))} = \overline{\mathcal{O}^\sigma(A)}$. So to show that $\bigl(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}, \sigma\bigr)$ is minimal, it suffices to verify that $\pi^*_\beta(1) \in \overline{\mathcal{O}^\sigma(A)}$. If on the contrary $\pi^*_\beta(1) \notin \overline{\mathcal{O}^\sigma(A)}$, then $\overline{\mathcal{O}^\sigma(A)}$ is a closed subset of $\mathcal{S}_\beta$ disjoint from $\pi^*_\beta(1)$, which is the maximal element in $\mathcal{S}_\beta$ (see (3.6) and (3.9)). Hence, from the fact that $\lim_{\gamma\nearrow\beta}\pi^*_\gamma(1)=\pi^*_\beta(1)$ (see Proposition 3.9 (xiv) and (3.8)), there exists $\gamma\in(1,\beta)$ with $\sigma^n(A)\preceq\pi^*_\gamma(1)$ for all $n\in\mathbb{N}_0$. Hence by (3.9), $A\in\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}\cap\mathcal{S}_\gamma$, which contradicts the assumption that β is emergent.

(iii) implies (i): Assume that β is non-emergent. By Definition 5.1 there exists $\gamma\in(1,\beta)$ and some $B\in\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}\cap\mathcal{S}_\gamma$. Hence the closure of $\mathcal{O}^\sigma(B)$ is contained in $\mathcal{S}_\gamma$, and therefore $\big(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))},\sigma\big)$ is not minimal. □

The following straightforward consequence of Proposition 5.3 will be useful in the sequel, in particular for the proofs of the Individual TPO and Joint TPO theorems.

Corollary 5.4. *Simple beta-numbers are emergent, and non-simple beta-numbers are non-emergent.*

Proof. Let $\beta>1$ be a simple beta-number. By Definition 3.10 (i) and Proposition 3.9 (i), $\pi^*_\beta(1)$ is a periodic sequence, so $\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}=\mathcal{O}^\sigma(\pi^*_\beta(1))$, and $\big(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))},\sigma\big)$ is minimal, therefore Proposition 5.3 implies that β is emergent.

Now let $\beta>1$ be a non-simple beta-number. By Definition 3.10 (ii) and Proposition 3.9 (i), $\pi^*_\beta(1)$ is a preperiodic but non-periodic sequence, so $\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}=\mathcal{O}^\sigma(\pi^*_\beta(1))$, and $\big(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))},\sigma\big)$ is not minimal, therefore Proposition 5.3 implies that β is non-emergent. □

Example 5.5. As a specific example of an emergent parameter that is not a simple beta-number, let $F_0=0$, $F_1=01$, and $F_{n+2}=F_{n+1}F_n$, $n\in\mathbb{N}_0$. The *Fibonacci word* F is then defined (see e.g. [Py02]) as the sequence $F:=0100101001\ldots$, whose prefixes are the F_n's, and is a Sturmian word (see e.g. [Py02]) of parameter $(3-\sqrt{5})/2$. There exists $\beta\in(1,2)$ such that $\pi_\beta(1)=1F$, i.e., the concatenation of 1 and F (cf. e.g. [CK04, p. 404]). Clearly, β is not a simple beta-number. Since $\big(\overline{\mathcal{O}^\sigma(1F)},\sigma\big)$ is minimal (cf. [CK04, pp. 397–398]), β is emergent.

Lemma 5.6. *If $\beta>1$ is emergent then the restriction $U_\beta|_{\overline{\mathcal{O}^*_\beta(1)}}$ is continuous, and the dynamical system $\big(\overline{\mathcal{O}^*_\beta(1)},U_\beta\big)$ is minimal.*

Proof. Since β is emergent, then $0\notin\overline{\mathcal{O}^*_\beta(1)}$ by Proposition 5.3. Write $\delta:=d\big(0,\overline{\mathcal{O}^*_\beta(1)}\big)$. By (3.2) and (3.5), $\overline{\mathcal{O}^*_\beta(1)}\cap(y,y+\delta/\beta)=\emptyset$ for each $y\in D_\beta$. So for each pair of x, $y\in\overline{\mathcal{O}^*_\beta(1)}$ with $|x-y|<\delta/\beta$, we have $(x,y)\cap D_\beta=\emptyset$ and hence $U_\beta(x)-U_\beta(y)=\beta(x-y)$. The first part follows.

Let $Y\subseteq\overline{\mathcal{O}^*_\beta(1)}$ be an arbitrary non-empty closed subset of $\overline{\mathcal{O}^*_\beta(1)}$ with $U_\beta(Y)=Y$. If $1\in Y$, then $Y=\overline{\mathcal{O}^*_\beta(1)}$. So to show that $\big(\overline{\mathcal{O}^*_\beta(1)},U_\beta\big)$ is minimal, it suffices to show that 1 must belong to Y. If on the contrary $1\notin Y$, then by Lemma 3.19 there exists $\gamma\in(1,\beta)$ such that $Y\subseteq H^\gamma_\beta$. But β is emergent, so $Y\subseteq\overline{\mathcal{O}^*_\beta(1)}\cap H^\gamma_\beta\subseteq Z_\beta$ by Proposition 5.3. By Lemma 3.26 (i), for each $y\in Y$, there exists $n\in\mathbb{N}$ with $U^n_\beta(y)=1$, which contradicts the assumption that $1\notin Y$. The lemma follows. □

By the following result, emergent parameters constitute a small subset of $(1,+\infty)$.

Corollary 5.7. *The emergent set E has zero Lebesgue measure, and is a meagre subset of* $(1,+\infty)$.

Proof. Suppose $\beta\in E$. By Proposition 5.3, $\big(\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))},\sigma\big)$ is minimal, so the fixed point $(0)^\infty$ does not belong to $\overline{\mathcal{O}^\sigma(\pi^*_\beta(1))}$, and thus the beta-shift $(\mathcal{S}_\beta,\sigma)$ is a specified system (see [Bl89, Proposition 4.5], [Sc97, Proposition 3.5]). So if Spec denotes the set of those $\beta>1$ such that $(\mathcal{S}_\beta,\sigma)$ is specified, then $E\subseteq$ Spec. But Spec is a meagre subset of $(1,+\infty)$ by [Sc97, Theorem B], and has zero Lebesgue measure by [Sc97, Theorem E], therefore E also has these properties. □

5.2. The critical set $\mathrm{Crit}^\alpha(\beta)$.

Definition 5.8. For $\beta>1$ and $\alpha\in(0,1]$, define the *critical set* $\mathrm{Crit}^\alpha(\beta)$ by

$$\mathrm{Crit}^\alpha(\beta):=\big\{\phi\in C^{0,\alpha}(I):\mathcal{O}^*_\beta(1)\text{ is a maximizing orbit for }(U_\beta,\phi)\big\}.\tag{5.1}$$

This naturally prompts an investigation into those invariant measures that are generated by the orbit of the point 1. It is convenient to first define the following notions in X_β:

Definition 5.9. For $\beta > 1$, define the *empirical measure* μ_n on X_β by

$$\mu_n \coloneqq \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i(\pi_\beta^*(1))}.$$

A measure $\mu \in \mathcal{M}(X_\beta, \sigma)$ is said to be *quasi-generated* by $\mathcal{O}^\sigma\bigl(\pi_\beta^*(1)\bigr)$ if it is an accumulation point of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. Let $\mathrm{QG}(\beta)$ denote the set of measures that are quasi-generated by $\mathcal{O}^\sigma(\pi_\beta^*(1))$ and let $\mathrm{CQG}(\beta)$ denote the convex hull of $\mathrm{QG}(\beta)$.

Clearly $\mathrm{QG}(\beta) \subseteq \mathcal{M}\bigl(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\bigr)$, and hence $\mathrm{CQG}(\beta) \subseteq \mathcal{M}\bigl(\overline{\mathcal{O}^\sigma(\pi_\beta^*(1))}, \sigma\bigr)$. The set $\mathrm{QG}(\beta)$ is known to be weak* closed (see [DGS76, Proposition 3.8] and [Gl03, p. 98]), hence $\mathrm{CQG}(\beta)$ is weak* closed.

The above notions lead to an alternative expression for the critical set $\mathrm{Crit}^\alpha(\beta)$:

Lemma 5.10. *Suppose $\beta > 1$ and $\alpha \in (0,1]$. Then*

$$\mathrm{Crit}^\alpha(\beta) = \bigl\{\phi \in C^{0,\alpha}(I) : H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)\bigr\}. \tag{5.2}$$

Moreover, $\mathrm{Crit}^\alpha(\beta)$ is a closed subset of $C^{0,\alpha}(I)$.

Proof. We first verify (5.2). By (5.1), Proposition 3.9 (v), and Proposition 3.27 (iii),

$$\phi \in \mathrm{Crit}^\alpha(\beta) \quad \text{if and only if} \quad \mathcal{O}^\sigma\bigl(\pi_\beta^*(1)\bigr) \text{ is } (\sigma|_{X_\beta}, \phi\circ h_\beta)\text{-maximizing.} \tag{5.3}$$

Assume that $\phi \in \mathrm{Crit}^\alpha(\beta)$. Then (5.3) implies that

$$\lim_{n\to+\infty} \frac{1}{n} S_n^\sigma(\phi\circ h_\beta)\bigl(\pi_\beta^*(1)\bigr) = Q(\sigma|_{X_\beta}, \phi\circ h_\beta),$$

and therefore $\mathrm{QG}(\beta) \subseteq \mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi\circ h_\beta)$. Hence $\mathrm{CQG}(\beta) \subseteq \mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi\circ h_\beta)$. By Proposition 3.27 (i) and (iii), it follows that $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$.

Conversely, assume that $\phi \notin \mathrm{Crit}^\alpha(\beta)$. By [Je19, Proposition 2.2],

$$\limsup_{n\to+\infty} \frac{1}{n} S_n^\sigma(\phi\circ h_\beta)\bigl(\pi_\beta^*(1)\bigr) \leqslant Q(\sigma|_{X_\beta}, \phi\circ h_\beta),$$

but the fact that $\phi \notin \mathrm{Crit}^\alpha(\beta)$ implies that there exists a subsequence $n_k \nearrow +\infty$ such that

$$\lim_{k\to+\infty} \frac{1}{n_k} S_{n_k}^\sigma(\phi\circ h_\beta)\bigl(\pi_\beta^*(1)\bigr) < Q(\sigma|_{X_\beta}, \phi\circ h_\beta).$$

Hence any accumulation point of the sequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$, which belongs to $\mathrm{QG}(\beta)$ by Definition 5.9, cannot belong to $\mathcal{M}_{\max}(\sigma|_{X_\beta}, \phi\circ h_\beta)$. By Proposition 3.27 (i) and (iii), it follows that $H_\beta(\mathrm{CQG}(\beta))$ is not contained in $\mathcal{M}_{\max}(U_\beta, \phi)$.

We next prove that $\mathrm{Crit}^\alpha(\beta)$ is a closed subset of $C^{0,\alpha}(I)$. If $\phi_n \in \mathrm{Crit}^\alpha(\beta)$, then

$$\int \phi_n \,\mathrm{d}\mu = Q(U_\beta, \phi_n) \tag{5.4}$$

for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$, by (5.2). If $\phi_n \to \phi$ in $(C^{0,\alpha}(I), \|\cdot\|_{\alpha,I})$, then $\int \phi_n \,\mathrm{d}\mu \to \int \phi \,\mathrm{d}\mu$ for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$. But $Q(U_\beta, \cdot)$ is (1-Lipschitz) continuous, so $Q(U_\beta, \phi_n) \to Q(U_\beta, \phi)$, and therefore (5.4) implies that $\int \phi \, d\mu = Q(U_\beta, \phi)$ for all $\mu \in H_\beta(\mathrm{CQG}(\beta))$. So $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$. Hence $\phi \in \mathrm{Crit}^\alpha(\beta)$ by (5.2), and therefore $\mathrm{Crit}^\alpha(\beta)$ is closed. □

If $\beta > 1$ is emergent, the following lemma gives another characterisation of $\mathrm{Crit}^\alpha(\beta)$.

Lemma 5.11. *Assume that $\beta > 1$ is emergent, $\alpha \in (0,1]$, and $\phi \in C^{0,\alpha}(I)$. Then the following are equivalent:*

(i) $\phi \in \mathrm{Crit}^\alpha(\beta)$.

(ii) $H_\beta(\mathrm{CQG}(\beta)) \subseteq \mathcal{M}_{\max}(U_\beta, \phi)$.
(iii) $1 \in \operatorname{supp}\mu$ *for some* $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$.

Proof. By Lemma 5.10, (i) is equivalent to (ii), so it suffices to prove that (i) is equivalent to (iii). For this, first assume that $\phi \in \mathrm{Crit}^\alpha(\beta)$, so that $\mathcal{O}^*_\beta(1)$ is (U_β, ϕ)-maximizing. Suppose μ is any accumulation point of $\left\{\frac{1}{n}\sum_{i=0}^{n-1}\delta_{U^i_\beta(1)}\right\}_{n\in\mathbb{N}}$. Note that $\overline{\mathcal{O}^*_\beta(1)}$ is compact, $U_\beta\big(\overline{\mathcal{O}^*_\beta(1)}\big) = \overline{\mathcal{O}^*_\beta(1)}$, and $U_\beta|_{\overline{\mathcal{O}^*_\beta(1)}}$ is continuous (see Lemma 5.6). We obtain that $\operatorname{supp}\mu \subseteq \overline{\mathcal{O}^*_\beta(1)}$ and $\mu \in \mathcal{M}\big(\overline{\mathcal{O}^*_\beta(1)}, U_\beta|_{\overline{\mathcal{O}^*_\beta(1)}}\big)$ (see [Wal82, Theorem 6.9]). Hence μ is also U_β-invariant as a measure on I. In addition, $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ since $\mathcal{O}^*_\beta(1)$ is (U_β, ϕ)-maximizing. But β is emergent, so $\big(\overline{\mathcal{O}^*_\beta(1)}, U_\beta\big)$ is minimal by Lemma 5.6. Note that μ can be seen as a measure on $\overline{\mathcal{O}^*_\beta(1)}$ and it follows from [Ak93, p. 156] that $U_\beta(\operatorname{supp}\mu) = \operatorname{supp}\mu$. Hence $\operatorname{supp}\mu$ must equal $\overline{\mathcal{O}^*_\beta(1)}$, and in particular $1 \in \operatorname{supp}\mu$.

Conversely, if we assume that $1 \in \operatorname{supp}\mu$ for some $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$, then (4.45), Theorem 4.9, Theorem 4.12 (ii), and Lemma 4.2 (iii) together imply that $\mathcal{O}^*_\beta(1)$ is a maximizing orbit for (U_β, ϕ), and hence that $\phi \in \mathrm{Crit}^\alpha(\beta)$. ☐

Lemma 5.12. *Suppose* $\beta > 1$, $\alpha \in (0,1]$, *and* $\phi \in C^{0,\alpha}(I)$. *If* $1 \notin \operatorname{supp}\mu$ *for some* $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$, *then there exists* $\beta' \in (1,\beta)$ *such that* $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ *for all* $\gamma \in (\beta', \beta)$.

Proof. Suppose $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ and $1 \notin \operatorname{supp}\mu =: \mathcal{K}$. It follows that $\mu \in \mathcal{M}(I, T_\beta)$ by Proposition 3.12 (iii), and $T_\beta(\mathcal{K}) = \mathcal{K}$ by Lemma 3.13. Therefore, by Lemma 3.19, there exists $\beta' \in (1,\beta)$ such that $\mathcal{K} \subseteq H^\gamma_\beta$ for all $\gamma \in (\beta', \beta)$. In particular, μ can be considered as an element of $\mathcal{M}\big(H^\gamma_\beta, T_\beta\big)$ for all $\gamma \in (\beta', \beta)$, and thus $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ by (3.12), as required. ☐

If β is emergent, we have the following corollary:

Corollary 5.13. *Suppose* $\beta > 1$ *is emergent,* $\alpha \in (0,1]$, *and* $\phi \in C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$. *Then there exists* $\beta' \in (1,\beta)$ *such that* $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ *for all* $\gamma \in (\beta', \beta)$.

Proof. Proposition 3.27 (iii) implies $\mathcal{M}_{\max}(U_\beta, \phi) \neq \emptyset$. Since $\phi \notin \mathrm{Crit}^\alpha(\beta)$, Lemma 5.11 implies that if $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ then $1 \notin \operatorname{supp}\mu$. The corollary now follows from Lemma 5.12. ☐

5.3. **The regular set $\mathcal{R}^\alpha(\beta)$.** In this subsection we establish a key result, the Dense Regular Functions Theorem (Theorem 5.16), asserting the density of the so-called *regular set* in $C^{0,\alpha}(I)$.

Notation. For each $\beta > 1$, let $\mathrm{Sim}(\beta)$ denote the set of those simple beta-numbers contained in the interval $(1, \beta)$.

Definition 5.14. For each $\beta > 1$ and each $\alpha \in (0,1]$, define the *regular set* $\mathcal{R}^\alpha(\beta) \subseteq C^{0,\alpha}(I)$ by

$$\mathcal{R}^\alpha(\beta) := \left\{\phi \in C^{0,\alpha}(I) : \phi|_{H^\gamma_\beta} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H^\gamma_\beta}\big) \text{ for all } \gamma \in \mathrm{Sim}(\beta)\right\}.$$

To prove the Dense Regular Functions Theorem, we will first use that for each $\beta \in (1, +\infty)$, $\mathrm{Sim}(\beta)$ is dense in $(1,\beta)$ by [Pa60, Theorem 5]. Then, by Definitions 3.6 and 3.10, $\gamma \in \mathrm{Sim}(\beta)$ if and only if $\pi_\gamma(1)$ has finitely many non-zero terms and $\gamma \in (1,\beta)$. Note that the function i_1, as defined in Definition 3.7, is by Proposition 3.9 (xiii) a strictly increasing map from $(1,\beta)$ to $\mathcal{B}^{\mathbb{N}}$, hence $\mathrm{Sim}(\beta)$ is countable, and enumerated as $\mathrm{Sim}(\beta) = \{\gamma_1, \dots, \gamma_n, \dots\}$, say. Then for each $\phi \in C^{0,\alpha}(I)$, Theorem 2.8 can be used to obtain a sequence of functions $\{\phi_n\}_{n\in\mathbb{N}}$ in $C^{0,\alpha}(I)$ sufficiently close to ϕ such that $\phi_n|_{H^{\gamma_n}_\beta} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H^{\gamma_n}_\beta}\big)$ for each $n \in \mathbb{N}$, and it can be verified that $\{\phi_n\}_{n\in\mathbb{N}}$ converges to a function $\phi_\infty \in C^{0,\alpha}(I)$ approximating ϕ.

In the proof of Theorem 5.16 below, we will need to extend a Hölder function, defined on a subset of I, to the whole of I, without increasing its norm. The following lemma guarantees that this can be done.

Lemma 5.15. *For* $\alpha \in (0,1]$ *and* $\emptyset \neq \mathcal{K} \subseteq I$, *and* $\phi \in C^{0,\alpha}(\mathcal{K})$, *there exists* $\psi \in C^{0,\alpha}(I)$ *such that* $\psi|_{\mathcal{K}} = \phi$ *and* $\|\psi\|_{\alpha,I} = \|\phi\|_{\alpha,\mathcal{K}}$.

Proof. This follows immediately from the McShane extension theorem ([Wea18, Theorem 1.33]). □

Now we are ready to state and prove Theorem 5.16:

Theorem 5.16 (Dense Regular Functions). *For all $\beta > 1$, $\alpha \in (0,1]$, the regular set $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$.*

Proof. Let $\varepsilon > 0$ and $\phi \in C^{0,\alpha}(I)$ be arbitrary. Since $\mathrm{Sim}(\beta)$ is countable, we write

$$\mathrm{Sim}(\beta) = \{\gamma_1, \dots, \gamma_n, \dots\}.$$

We will recursively construct a sequence of functions $\{\phi_n\}_{n\in\mathbb{N}}$ in $C^{0,\alpha}(I)$, and two sequences of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{e_n\}_{n\in\mathbb{N}}$ below:

Base step. Define $\phi_0 \coloneqq \phi$ and $\delta_0 \coloneqq 1$.

Recursive step. For $n \in \mathbb{N}$, assume that $\phi_0, \dots, \phi_{n-1}, \delta_0, \dots, \delta_{n-1}$ are defined. Define

$$e_n \coloneqq \min\bigl\{2^{-n}\varepsilon,\, 2^{-n}\delta_0,\, 2^{-n+1}\delta_1,\, \dots,\, 2^{-1}\delta_{n-1}\bigr\}. \tag{5.5}$$

Since γ_n is a simple beta-number, Proposition 3.21 gives that $T_\beta|_{H_\beta^{\gamma_n}}$ is Lipschitz continuous, open, and distance-expanding, and then Contreras' Individual TPO theorem (Theorem 2.8) guarantees that there exists $\psi_n \in C^{0,\alpha}\bigl(H_\beta^{\gamma_n}\bigr)$ satisfying

$$\psi_n \in \mathrm{Lock}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_n}}\bigr) \quad \text{and} \quad \bigl\|\phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n\bigr\|_{\alpha, H_\beta^{\gamma_n}} < e_n.$$

Applying Lemma 5.15, for the subset $H_\beta^{\gamma_n} \subseteq I$ and function $\phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n$, we obtain an α-Hölder extension function Φ_n defined on all of I, and with the same α-Hölder norm, in other words, there exists $\Phi_n \in C^{0,\alpha}(I)$ satisfying $\Phi_n|_{H_\beta^{\gamma_n}} = \phi_{n-1}|_{H_\beta^{\gamma_n}} - \psi_n$ and $\|\Phi_n\|_{\alpha,I} < e_n$. Defining $\phi_n \coloneqq \phi_{n-1} - \Phi_n$ then gives

$$\phi_n|_{H_\beta^{\gamma_n}} = \psi_n \quad \text{and} \quad \|\phi_{n-1} - \phi_n\|_{\alpha,I} < e_n. \tag{5.6}$$

Finally, δ_n is defined to be any value such that

$$\bigl\{\Phi \in C^{0,\alpha}\bigl(H_\beta^{\gamma_n}\bigr) : \bigl\|\Phi - \phi_n|_{H_\beta^{\gamma_n}}\bigr\|_{\alpha, H_\beta^{\gamma_n}} < \delta_n\bigr\} \subseteq \mathrm{Lock}^\alpha\bigl(T_\beta|_{H_\beta^{\gamma_n}}\bigr), \tag{5.7}$$

where again Theorem 2.8 and Proposition 3.21 guarantee that such a δ_n exists. Since each of e_n, ϕ_n, and δ_n have been defined, the recursive step is complete.

By (5.6) and (5.5), $\{\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to some $\phi_\infty \in C^{0,\alpha}(I)$, and

$$\|\phi - \phi_\infty\|_{\alpha,I} \leqslant \sum_{n=1}^{+\infty} \|\phi_{n-1} - \phi_n\|_{\alpha,I} < \sum_{n=1}^{+\infty} e_n \leqslant \varepsilon. \tag{5.8}$$

For each $n \in \mathbb{N}$, from (5.5) it follows that $\sum_{i=n+1}^{+\infty} e_i \leqslant \delta_n$ and then

$$\bigl\|\phi_n|_{H_\beta^{\gamma_n}} - \phi_\infty|_{H_\beta^{\gamma_n}}\bigr\|_{\alpha, H_\beta^{\gamma_n}} \leqslant \|\phi_n - \phi_\infty\|_{\alpha,I} \leqslant \sum_{i=n}^{+\infty} \|\phi_i - \phi_{i+1}\|_{\alpha,I} < \sum_{i=n+1}^{+\infty} e_i \leqslant \delta_n. \tag{5.9}$$

But (5.7) and (5.9) imply that $\phi_\infty|_{H_\beta^{\gamma_n}} \in \mathrm{Lock}^\alpha(T_\beta|_{H_\beta^{\gamma_n}})$ for all $n \in \mathbb{N}$, in other words, $\phi_\infty \in \mathcal{R}^\alpha(\beta)$. But $\varepsilon > 0$ was arbitrary, so the result follows. □

Corollary 5.17. *For all $\beta > 1$ and $\alpha \in (0,1]$, the set $\mathcal{R}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$ is a dense subset of $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^\alpha(\beta)$.*

Proof. This follows immediately from the fact that $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$ by Theorem 5.16, and the fact that $\mathrm{Crit}^\alpha(\beta)$ is closed in $C^{0,\alpha}(I)$ by Lemma 5.10. □

5.4. **A structural theorem for emergent parameters.** In this subsection we shall be primarily concerned with those parameters β that are emergent. For such β it remains an open question as to whether U_β has the typical periodic optimization property, nevertheless we shall establish a structural theorem (Theorem 5.22) that identifies the critical set $\mathrm{Crit}^\alpha(\beta)$ as the only possible obstacle to TPO. We first require:

Corollary 5.18. *If $\beta > 1$ is emergent and $\alpha \in (0,1]$, then*

$$\mathcal{R}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(U_\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta). \tag{5.10}$$

Proof. Suppose $\phi \in \mathcal{R}^\alpha(\beta) \smallsetminus \mathrm{Crit}^\alpha(\beta)$. By Corollary 5.13, there is a simple beta-number $\gamma_0 \in (1,\beta)$ such that

$$Q_{\beta,\gamma_0}(\phi) = Q(U_\beta, \phi). \tag{5.11}$$

The fact that $\phi \in \mathcal{R}^\alpha(\beta)$ implies that

$$\phi|_{H_\beta^{\gamma_0}} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big) \subseteq \mathscr{P}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big).$$

So there exists a periodic measure $\mu \in \mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$ (see Proposition 3.12 (iii)) such that

$$\int \phi \,\mathrm{d}\mu = Q_{\beta,\gamma_0}(\phi). \tag{5.12}$$

So from (5.11) and (5.12) we see that the periodic measure μ satisfies $\int \phi \,\mathrm{d}\mu = Q(U_\beta, \phi)$, and therefore $\phi \in \mathscr{P}^\alpha(U_\beta)$. But $\phi \notin \mathrm{Crit}^\alpha(\beta)$, so (5.10) follows. $\square$

In the sequel, it will be convenient to articulate the relationship between the set of functions where at least one maximizing measure is periodic, and its subset consisting of those functions whose maximizing measure is unique, periodic, and stably maximizing under perturbations. For this, recall that $\mathscr{P}^\alpha(U_\beta)$ denotes the set of those $\phi \in C^{0,\alpha}(I)$ with a (U_β, ϕ)-maximizing measure supported on a periodic orbit of U_β, and that $\mathrm{Lock}^\alpha(U_\beta)$ consists of those functions $\phi \in \mathscr{P}^\alpha(U_\beta)$ with $\operatorname{card} \mathcal{M}_{\max}(U_\beta, \phi) = 1$ and $\mathcal{M}_{\max}(U_\beta, \phi) = \mathcal{M}_{\max}(U_\beta, \psi)$ for all $\psi \in C^{0,\alpha}(I)$ sufficiently close to ϕ in $C^{0,\alpha}(I)$. We wish to show that $\mathrm{Lock}^\alpha(U_\beta)$ is dense in $\mathscr{P}^\alpha(U_\beta)$ (see Theorem 5.20 below). A statement analogous to this appeared as Remark 4.5 in [YH99] for maps with hyperbolicity, and as Proposition 1 in the unpublished note [BZ15] for continuous maps. Our method of proof, using ideas from [BZ15], begins with the following lemma.

Lemma 5.19. *Suppose $\beta > 1$, $\alpha \in (0,1]$, and let $\mu \in \mathcal{M}(I, U_\beta)$ be supported on a periodic orbit $\mathcal{O}^*$ of U_β. Then there exists $C_\mu > 0$ such that for all $\nu \in \mathcal{M}(I, U_\beta)$ and $\phi \in C^{0,\alpha}(I)$,*

$$\int_I \phi \,\mathrm{d}\nu \leqslant \int_I \phi \,\mathrm{d}\mu + C_\mu |\phi|_{\alpha,I} \int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu. \tag{5.13}$$

Proof. Let us write $n := \operatorname{card} \mathcal{O}^*$.

Case I. If $n = 1$ then $\mathcal{O}^* = \{y\}$ for some $y \in I$, and (5.13) holds with $C_\mu = 1$ because $\int_I \phi \,\mathrm{d}\nu \leqslant \int_I (\phi(y) + |\phi|_{\alpha,I} d(\cdot, y)^\alpha) \,\mathrm{d}\nu = \int_I \phi \,\mathrm{d}\mu + |\phi|_{\alpha,I} \int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu$.

Case II. If $n > 1$ then by ergodic decomposition, it suffices to prove (5.13) for every ergodic $\nu \in \mathcal{M}(I, U_\beta)$. Fixing an arbitrary ergodic $\nu \in \mathcal{M}(I, U_\beta)$, the ergodic theorem implies that there exists $a \in I$ with

$$\int_I \phi \,\mathrm{d}\nu = \lim_{k \to +\infty} \frac{1}{k} S_k^{U_\beta} \phi(a) \quad \text{and} \tag{5.14}$$

$$\int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu = \lim_{k \to +\infty} \frac{1}{k} \sum_{i=0}^{k-1} d\big(U_\beta^i(a), \mathcal{O}^*\big)^\alpha. \tag{5.15}$$

Claim. There exists $C_\mu > 0$ and a sequence $\underline{y} = \{y_i\}_{i=-1}^{+\infty}$ with entries from $\mathcal{O}^*$ such that

$$\lim_{k \to +\infty} \frac{1}{k} \operatorname{card}\{i \in [0, k-1] \cap \mathbb{N}_0 : y_i = y\} = \frac{1}{n} \quad \text{for each } y \in \mathcal{O}^* \text{ and} \tag{5.16}$$

$$\big|U_\beta^i(a) - y_i\big| \leqslant C_\mu^{1/\alpha} d\big(U_\beta^i(a), \mathcal{O}^*\big) \qquad \text{for each } i \in \mathbb{N}_0. \tag{5.17}$$

Note that a consequence of this Claim is, by (5.14), (5.17), (5.16), and (5.15), that

$$\begin{aligned}
\int_I \phi \,\mathrm{d}\nu &= \lim_{k\to+\infty} \frac{1}{k}\sum_{i=0}^{k-1} \phi\big(U_\beta^i(a)\big) \\
&\leqslant \lim_{k\to+\infty} \frac{1}{k}\sum_{i=0}^{k-1} \big(\phi(y_i) + |\phi|_{\alpha,I}\big|U_\beta^i(a) - y_i\big|^\alpha\big) \\
&\leqslant \lim_{k\to+\infty} \frac{1}{k}\sum_{i=0}^{k-1} \big(\phi(y_i) + C_\mu|\phi|_{\alpha,I} d\big(U_\beta^i(a), \mathcal{O}^*\big)^\alpha\big) \\
&= \frac{1}{n}\sum_{y\in\mathcal{O}^*} \phi(y) + C_\mu|\phi|_{\alpha,I} \lim_{k\to+\infty} \frac{1}{k}\sum_{i=0}^{k-1} d\big(U_\beta^i(a), \mathcal{O}^*\big)^\alpha \\
&= \int_I \phi\,\mathrm{d}\mu + C_\mu|\phi|_{\alpha,I}\int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu.
\end{aligned}$$

So the required inequality (5.13) will hold, and the lemma will follow.

Proof of Claim. Our discussion will be divided into 2 cases. Recall (cf. Subsection 1.5) that since $\operatorname{card}\mathcal{O}^* > 1$, the minimum interpoint distance is given by $\Delta(\mathcal{O}^*) = \min\{d(x,y) : x, y \in \mathcal{O}^*, x \neq y\}$.

Case 1. Assume that $\mathcal{O}^*$ is not the orbit of 1 under U_β. By Proposition 3.12 (i), $D_\beta \cap \mathcal{O}^* = \emptyset$. Set

$$\delta \coloneqq (1/2)\Delta(\mathcal{O}^*) \tag{5.18}$$

and $\varepsilon^* \coloneqq (1/2)\beta^{-n} d(\mathcal{O}^*, D_\beta)$, so that $U_\beta^k\big|_{B(\mathcal{O}^*,\varepsilon^*)}$ is continuous for all $0 \leqslant k \leqslant n-1$. For each $x \in \mathcal{O}^*$, there exists $\varepsilon_x \in (0, \varepsilon^*)$ such that $\big|U_\beta^i(x) - U_\beta^i(y)\big| < \delta$ for all $y \in (x - \varepsilon_x, x + \varepsilon_x)$ and $0 \leqslant i \leqslant n-1$. Moreover, if $\varepsilon \coloneqq \min\{\varepsilon_x : x \in \mathcal{O}^*\}$ then

$$\big|U_\beta^k(x) - U_\beta^k(y)\big| < \delta \tag{5.19}$$

for all $x \in \mathcal{O}^*$, $y \in B(x, \varepsilon)$, and $0 \leqslant k \leqslant n-1$.

The sequence $\underline{y}$ is constructed recursively as follows.

Base step. Choose an arbitrary $y_{-1} \in \mathcal{O}^*$, and mark y_{-1} as a bad point.

Recursive step. For some $t \in \mathbb{N}_0$, assume that $y_{-1}, y_0, \dots, y_{t-1}$ are defined.

If $d\big(U_\beta^t(a), \mathcal{O}^*\big) < \varepsilon$, choose $y_t \in \mathcal{O}^*$ such that $d\big(U_\beta^t(a), \mathcal{O}^*\big) = \big|U_\beta^t(a) - y_t\big|$, set $y_{t+i} \coloneqq U_\beta^i(y_t)$ for each $1 \leqslant i \leqslant n-1$, and mark $y_t, y_{t+1}, \dots, y_{t+n-1}$ as good points.

If $d\big(U_\beta^t(a), \mathcal{O}^*\big) \geqslant \varepsilon$, let p_t be the number of bad points in $\{y_{-1}, y_0, \dots, y_{t-1}\}$, then set $y_t \coloneqq U_\beta^{p_t}(y_{-1})$ and mark y_t as a bad point.

Note that the required (5.16) is immediate from the above construction. To prove (5.17), note that for each bad point y_i, $i \in \mathbb{N}_0$, we have $d\big(U_\beta^i(a), \mathcal{O}^*\big) \geqslant \varepsilon \geqslant \varepsilon d\big(U_\beta^i(a), y_i\big)$, and for each good point y_i, we obtain $d\big(U_\beta^i(a), \mathcal{O}^*\big) = d\big(U_\beta^i(a), y_i\big)$ by (5.18), (5.19), and our construction. Therefore (5.17) holds by choosing $C_\mu \coloneqq \max\{1, \varepsilon^{-\alpha}\}$, so Claim is proved for Case 1.

Case 2. Assume that $\mathcal{O}^*$ is the orbit of 1 under U_β. By Remark 3.11, β is a simple beta-number. By Proposition 3.12 (i), $D_\beta \cap \mathcal{O}_\beta^*(1) = U_\beta^{-1}(1) \cap \mathcal{O}^* = \big\{U_\beta^{n-1}(1)\big\}$ and $U_\beta^{-1}(0) = \{0\}$. Set

$$\delta' \coloneqq \min\{(1/2)\Delta(\mathcal{O}^*), (1/2)d(0, \mathcal{O}^*)\}. \tag{5.20}$$

As $U_\beta^{-1}(0) = \{0\}$, then $0 \notin \mathcal{O}^*$, so $\delta' > 0$. For each $0 \leqslant i \leqslant n-1$ and $z_i \coloneqq U_\beta^i(1)$, we claim that there exists $\varepsilon_i > 0$ satisfying the following properties:

(1) $\big|U_\beta^j(z_i) - U_\beta^j(y)\big| < \delta'$ for all $y \in (z_i - \varepsilon_i, z_i]$, $0 \leqslant j \leqslant n-1$.

(2) $\big|U_\beta^j(z_i) - U_\beta^j(y)\big| < \delta'$ for all $y \in (z_i, z_i + \varepsilon_i)$, $0 \leqslant j \leqslant n-i-1$.

(3) $\left|U_\beta^j(y)\right| < \delta'$ for all $y \in (z_i, z_i + \varepsilon_i)$, $n - i \leqslant j \leqslant n - 1$.

Indeed, by Lemma 3.4 (i), $\lim_{y \nearrow x} U_\beta^j(y) = U_\beta^j(x)^-$ for each $x \in (0, 1]$ and each $j \in \mathbb{N}$. Thus, there exists $\varepsilon_{i,1} > 0$ satisfying property (1).

Now define $A_i := \emptyset$ when $i = n - 1$ and $A_i := \left\{z_i, \dots, U_\beta^{n-i-2}(z_i)\right\}$ when $i < n - 1$. Note that for every $0 \leqslant i \leqslant n - 1$, $U_\beta = T_\beta$ in a neighbourhood of A_i (see Remark 3.3) since $A_i \cap D_\beta = \emptyset$, so if $0 \leqslant j \leqslant n - i - 1$ then by Lemma 3.4 (i), we have $\lim_{y \searrow z_i} U_\beta^j(y) = U_\beta^j(z_i)^+$. Hence, there exists $\varepsilon_{i,2} > 0$ satisfying property (2).

Since $U_\beta^{n-1}(1) \in D_\beta$, by (3.2), we get that $\lim_{y \searrow z_i} U_\beta^j(y) = 0$ for each $n - i \leqslant j \leqslant n - 1$. Thus, there exists $\varepsilon_{i,3} > 0$ satisfying property (3).

Defining $\varepsilon_i := \min\{\varepsilon_{i,1}, \varepsilon_{i,2}, \varepsilon_{i,3}\}$, we see that ε_i satisfies properties (1), (2), and (3).

Now define $\varepsilon' := \min\{\varepsilon_i : 0 \leqslant i \leqslant n - 1\}$, and construct $\underline{y}$ as in Case 1, except that ε and δ are replaced, respectively, by ε' and δ'. Then (5.16) holds immediately, while for each bad point y_i, $i \in \mathbb{N}_0$, we have $d\big(U_\beta^i(a), \mathcal{O}^*\big) \geqslant \varepsilon' \geqslant \varepsilon' d\big(U_\beta^i(a), y_i\big)$, and for each good point y_i, by (5.20) and properties (1), (2), and (3), either $d\big(U_\beta^i(a), \mathcal{O}^*\big) = d\big(U_\beta^i(a), y_i\big)$ or $d\big(U_\beta^i(a), \mathcal{O}^*\big) \geqslant \delta' \geqslant \delta' d\big(U_\beta^i(a), y_i\big)$. Hence (5.17) holds if we take $C_\mu := \min\{(\varepsilon')^{-\alpha}, (\delta')^{-\alpha}, 1\}$, so Claim is proved for Case 2. □

Having established Lemma 5.19, we can now prove the following Theorem 5.20.

Theorem 5.20. *If $\beta > 1$ and $\alpha \in (0, 1]$, then the set* $\operatorname{Lock}^\alpha(U_\beta)$ *is an open and dense subset of* $\mathscr{P}^\alpha(U_\beta)$.

Proof. If $\phi \in \mathscr{P}^\alpha(U_\beta)$, choose $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$ supported on a periodic orbit $\mathcal{O}^*$ of U_β, and for each $t > 0$ define

$$\phi_t := \phi - t d(\cdot, \mathcal{O}^*)^\alpha \in C^{0,\alpha}(I).$$

Clearly ϕ_t belongs to $C^{0,\alpha}(I)$, and converges to ϕ as $t \to 0$. By Lemma 5.19, if $t > 0$, $\nu \in \mathcal{M}(I, U_\beta) \smallsetminus \{\mu\}$, and $\psi \in C^{0,\alpha}(I)$ with $\|\psi\|_{\alpha,I} < t/C_\mu$, then

$$\begin{aligned}\int_I (\phi_t + \psi) \,\mathrm{d}\nu &= \int_I \phi \,\mathrm{d}\nu + \int_I \psi \,\mathrm{d}\nu - t \int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu \leqslant \int_I \phi \,\mathrm{d}\mu + \int_I \psi \,\mathrm{d}\mu + (C_\mu |\psi|_{\alpha,I} - t) \int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu \\ &= \int_I (\phi_t + \psi) \,\mathrm{d}\mu + (C_\mu |\psi|_{\alpha,I} - t) \int_I d(\cdot, \mathcal{O}^*)^\alpha \,\mathrm{d}\nu < \int_I (\phi_t + \psi) \,\mathrm{d}\mu,\end{aligned}$$

so $\mathcal{M}_{\max}\big(U_\beta, \phi_t + \psi\big) = \{\mu\}$, and consequently, $\phi_t \in \operatorname{Lock}^\alpha(U_\beta)$. Hence, $\operatorname{Lock}^\alpha(U_\beta)$ is dense in $\mathscr{P}^\alpha(U_\beta)$. Moreover, from their definitions, $\operatorname{Lock}^\alpha(U_\beta)$ is an open subset of $\mathscr{P}^\alpha(U_\beta)$, therefore the theorem is established. □

Remark 5.21. As mentioned in Section 1, for certain parameters β there exist continuous functions ϕ without a (T_β, ϕ)-maximizing measure (e.g. for $\beta = 2$, the function $\phi(x) := x$ has no maximizing measure; indeed, there is an open neighbourhood of ϕ in $C^{0,1}(I)$ consisting of functions with no maximizing measure); this absence of a maximizing measure is due to the lack of compactness of $\mathcal{M}(I, T_\beta)$. More generally, for β a simple beta-number and $\alpha \in (0, 1]$, the above proof of Theorem 5.20 can be used to show that there is a non-empty open subset of $C^{0,\alpha}(I)$ consisting of functions with no maximizing measure. Specifically, the function $\phi := -d\big(\cdot, \mathcal{O}_\beta^*(1)\big)^\alpha$ has the locking property (with respect to U_β) in $C^{0,\alpha}(I)$, with the unique (U_β, ϕ)-maximizing measure $\mu_{\mathcal{O}_\beta^*(1)}$; since $\mu_{\mathcal{O}_\beta^*(1)}$ is not T_β-invariant, but $Q(T_\beta, \phi) = Q(U_\beta, \phi)$ by Proposition 3.28 (ii), and $\mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$ by Proposition 3.12 (iii), any function ψ sufficiently close to ϕ has no (T_β, ψ)-maximizing measure. The phenomenon of absence of maximizing measures when β is a simple beta-number means that for T_β (as distinct from U_β), the TPO property does not hold in the whole of $C^{0,\alpha}(I)$, for the straightforward reason that *optimization itself* is not a typical property in $C^{0,\alpha}(I)$.

We can now prove a structural theorem for emergent parameters β:

Theorem 5.22 (Structural theorem for emergent parameters). *If $\beta > 1$ is emergent and $\alpha \in (0, 1]$, then $C^{0,\alpha}(I)$ is equal to the union of the critical set $\mathrm{Crit}^{\alpha}(\beta)$ and the closure of the open set $\mathrm{Lock}^{\alpha}(U_\beta)$.*

Proof. Fix an arbitrary $\beta > 1$ that is emergent, and $\alpha \in (0, 1]$. We wish to show that $\mathrm{Lock}^{\alpha}(U_\beta) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$ is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$. By Theorem 5.20 and the fact that $\mathrm{Crit}^{\alpha}(\beta)$ is closed (see Lemma 5.10), it suffices to prove that $\mathscr{P}^{\alpha}(U_\beta) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$ is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$. The set $\mathcal{R}^{\alpha}(\beta) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$ is a subset of $\mathscr{P}^{\alpha}(U_\beta) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$ by Corollary 5.18, and is dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$, by Corollary 5.17. Therefore, $\mathscr{P}^{\alpha}(U_\beta) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$ is itself dense in $C^{0,\alpha}(I) \smallsetminus \mathrm{Crit}^{\alpha}(\beta)$, as required. □

5.5. **Proof of Individual TPO theorems.** In this subsection, we prove the individual typical periodic optimization theorems stated in Section 1, namely for generic β (Theorem D), for Lebesgue almost every β (Theorem E), whenever β is a beta-number (Theorem G), and whenever β is non-emergent (Theorem H).

Having considered emergent parameters β in Subsection 5.4, we begin with a number of results about non-emergent parameters:

Lemma 5.23. *Suppose $\beta > 1$ is non-emergent, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$. Then there exists $\beta' \in (1, \beta)$ such that $Q_{\beta,\beta'}(\phi) = Q(U_\beta, \phi)$.*

Proof. By Proposition 3.27 (iii) there exists $\mu \in \mathcal{M}_{\max}(U_\beta, \phi)$. Let us denote $\mathcal{K} := \operatorname{supp}\mu$.

If $0 \in \mathcal{K}$, then $\widetilde{\phi}^{+}(0) = 0 = Q\bigl(U_\beta, \widetilde{\phi}^{+}\bigr)$ or $\widetilde{\phi}^{-}(0) = 0 = Q\bigl(U_\beta, \widetilde{\phi}^{-}\bigr)$ by Theorems 4.12 (i) and 4.9 (ii), so $\delta_0 \in \mathcal{M}_{\max}(U_\beta, \phi)$ (by (4.45), (4.46), and Lemma 4.2 (i)), and therefore $Q_{\beta,\gamma}(\phi) = Q(U_\beta, \phi)$ for every $\gamma \in (1, \beta)$.

If $0 \notin \mathcal{K}$ then $U_\beta(\mathcal{K}) = \mathcal{K}$ by Lemma 3.13. Let us assume, for a contradiction, that the result is false, i.e., that

$$Q_{\beta,\gamma}(\phi) < Q(U_\beta, \phi) \quad \text{for all } \gamma \in (1, \beta). \tag{5.21}$$

Lemma 5.12 then implies that $1 \in \mathcal{K}$. Since β is non-emergent, Proposition 5.3 implies that there exists $\gamma \in (1, \beta)$ such that $\overline{\mathcal{O}_\beta^*(1)} \cap H_\beta^\gamma$ is not a subset of Z_β, where we recall from (3.15) that $Z_\beta = \bigl\{x \in I : \pi_\beta(x) \neq \pi_\beta^*(x)\bigr\}$. But $1 \in \mathcal{K}$, so $\overline{\mathcal{O}_\beta^*(1)} \subseteq \mathcal{K}$, and therefore $\mathcal{K} \cap H_\beta^\gamma$ is not a subset of Z_β, in other words, there exists $x \in \bigl(\mathcal{K} \cap H_\beta^\gamma\bigr) \smallsetminus Z_\beta$.

By Lemma 3.26 (iii), the orbit $\mathcal{O}_\beta(x)$ is equal to $\mathcal{O}_\beta^*(x)$, and it is contained in $\mathcal{K} \cap H_\beta^\gamma$ since $U_\beta(\mathcal{K}) = \mathcal{K}$ and $T_\beta\bigl(H_\beta^\gamma\bigr) \subseteq H_\beta^\gamma$. By Theorem 4.12 (i), $\widetilde{\phi}^{+}|_{\mathcal{O}_\beta(x)} \equiv 0$ or $\widetilde{\phi}^{-}|_{\mathcal{O}_\beta(x)} \equiv 0$. In other words, $\mathcal{O}_\beta(x)$ is contained in either $\bigl(\widetilde{\phi}^{-}\bigr)^{-1}(0)$ or $\bigl(\widetilde{\phi}^{+}\bigr)^{-1}(0)$. Since $\widetilde{\phi}^{-} \leqslant 0$ and $\widetilde{\phi}^{+} \leqslant 0$ (by Theorem 4.9 (ii)), Lemma 4.2 (iii) implies that $\mathcal{O}_\beta(x) = \mathcal{O}_\beta^*(x)$ is a (T_β, ϕ)-maximizing orbit and a (U_β, ϕ)-maximizing orbit (by Proposition 3.28 (ii)), so in particular,

$$\lim_{n\to+\infty} \frac{1}{n} S_n\phi(x) = Q(U_\beta, \phi). \tag{5.22}$$

Now $\mathcal{O}_\beta(x) \subseteq H_\beta^\gamma$ and $T_\beta|_{H_\beta^\gamma}$ is continuous (see Proposition 3.21 (i)), so by [Je19, Proposition 2.2], the corresponding time average is bounded above by the ergodic supremum, in other words,

$$Q_{\beta,\gamma}(\phi) \geqslant \lim_{n\to+\infty} \frac{1}{n} S_n\phi(x). \tag{5.23}$$

Now as an immediate consequence of (3.12), we have $Q(U_\beta, \phi) \geqslant Q_{\beta,\gamma}(\phi)$. So combining this inequality with (5.22) and (5.23) gives $Q(U_\beta, \phi) = Q_{\beta,\gamma}(\phi)$, which gives the required contradiction to (5.21). The lemma follows. □

Corollary 5.24. *Suppose $\beta > 1$ is non-emergent, $\alpha \in (0, 1]$, and $\phi \in C^{0,\alpha}(I)$. Then there exists $\beta' \in (1, \beta)$ such that*

$$Q_{\beta,\gamma}(\phi) = Q\bigl(U_\beta, \phi\bigr) \qquad \textit{for all } \gamma \in [\beta', \beta). \tag{5.24}$$

Proof. Let β' be as in Lemma 5.23. If $\gamma \in [\beta', \beta)$ then

$$Q_{\beta,\beta'}(\phi) \leqslant Q_{\beta,\gamma}(\phi) \leqslant Q(U_\beta, \phi), \tag{5.25}$$

an immediate consequence of (3.12), since $H_\beta^{\beta'} \subseteq H_\beta^\gamma \subseteq I$. But $Q(U_\beta, \phi) = Q_{\beta,\beta'}(\phi)$, by Lemma 5.23, so (5.25) implies the required equality (5.24). □

The proof of the following result is similar to that of Corollary 5.18:

Corollary 5.25. *If $\beta > 1$ is non-emergent, then $\mathcal{R}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(U_\beta)$.*

Proof. Suppose $\phi \in \mathcal{R}^\alpha(\beta)$. By Corollary 5.24, there exists $\beta' \in (1, \beta)$ such that $Q_{\beta,\gamma}(\phi) = Q\big(U_\beta, \phi\big)$ for all $\gamma \in [\beta', \beta)$, so in particular there is a simple beta-number $\gamma_0 \in (1, \beta)$ such that

$$Q_{\beta,\gamma_0}(\phi) = Q(U_\beta, \phi). \tag{5.26}$$

The fact that $\phi \in \mathcal{R}^\alpha(\beta)$ implies that

$$\phi|_{H_\beta^{\gamma_0}} \in \mathrm{Lock}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big) \subseteq \mathscr{P}^\alpha\big(T_\beta|_{H_\beta^{\gamma_0}}\big).$$

So there exists a periodic measure $\mu \in \mathcal{M}(I, T_\beta) \subseteq \mathcal{M}(I, U_\beta)$ (see Proposition 3.12 (iii)) such that

$$\int \phi \, \mathrm{d}\mu = Q_{\beta,\gamma_0}(\phi). \tag{5.27}$$

Thus from (5.26) and (5.27) we see that the periodic measure μ satisfies $\int \phi \, \mathrm{d}\mu = Q(U_\beta, \phi)$. Therefore $\phi \in \mathscr{P}^\alpha(U_\beta)$, as required. □

We are now in a position to prove our Individual TPO theorems. We establish the following slightly stronger version of Theorem H (which in particular implies Theorem H):

Theorem H′ (Individual TPO for non-emergent parameters). *Fix $\alpha \in (0, 1]$. If $\beta > 1$ is non-emergent, then both $\mathrm{Lock}^\alpha(T_\beta)$ and $\mathrm{Lock}^\alpha(U_\beta)$ are open and dense subsets of $C^{0,\alpha}(I)$.*

Proof. Let $\beta > 1$ be non-emergent. By Corollary 5.4, β cannot be a simple beta-number. So Theorem 3.12 (iv) implies that $\mathrm{Per}(T_\beta) = \mathrm{Per}(U_\beta)$ and $\mathcal{M}(I, T_\beta) = \mathcal{M}(I, U_\beta)$. This implies that $\mathrm{Lock}^\alpha(T_\beta) = \mathrm{Lock}^\alpha(U_\beta)$. Thus it suffices to show that $\mathrm{Lock}^\alpha(U_\beta)$ is an open dense subset of $C^{0,\alpha}(I)$.

Now $\mathrm{Lock}^\alpha(U_\beta)$ is by definition an open subset of $C^{0,\alpha}(I)$, and Theorem 5.20 asserts that $\mathrm{Lock}^\alpha(U_\beta)$ is dense in $\mathscr{P}^\alpha(U_\beta)$, so it suffices to prove that $\mathscr{P}^\alpha(U_\beta)$ is dense in $C^{0,\alpha}(I)$. Since $\beta > 1$ is non-emergent, Corollary 5.25 gives $\mathcal{R}^\alpha(\beta) \subseteq \mathscr{P}^\alpha(U_\beta)$, and $\mathcal{R}^\alpha(\beta)$ is dense in $C^{0,\alpha}(I)$ by Theorem 5.16. Therefore, it follows that $\mathscr{P}^\alpha(U_\beta)$ is dense in $C^{0,\alpha}(I)$, as required. □

The following is a slightly stronger version of Theorem D, which in particular implies Theorem D.

Theorem D′ (Individual TPO for generic parameters β). *Fix $\alpha \in (0, 1]$. For a residual set of values $\beta > 1$, $\mathrm{Lock}^\alpha(T_\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Proof. By Corollary 5.7, the set $(1, +\infty) \smallsetminus E$ of non-emergent parameters is a residual subset of $(1, +\infty)$. By Theorem H′, if $\beta \in (1, +\infty) \smallsetminus E$ then $\mathrm{Lock}^\alpha(T_\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$, so the result follows. □

The following is a slightly stronger version of Theorem E, which in particular implies Theorem E.

Theorem E′ (Individual TPO for almost every parameter β). *Fix $\alpha \in (0, 1]$. For Lebesgue almost every $\beta > 1$, $\mathrm{Lock}^\alpha(T_\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$.*

Proof. By Corollary 5.7, the set $(1, +\infty) \smallsetminus E$ of non-emergent parameters has full Lebesgue measure. By Theorem H′, if $\beta \in (1, +\infty) \smallsetminus E$ then $\mathrm{Lock}^\alpha(T_\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$, so the result follows. □

We now prove a slightly stronger version of Theorem G, which in particular implies Theorem G.

Theorem G′ (Individual TPO for beta-numbers). *Fix* $\alpha \in (0,1]$. *If* $\beta > 1$ *is a beta-number, then* $\operatorname{Lock}^\alpha(U_\beta)$ *is an open and dense subset of* $C^{0,\alpha}(I)$.

Proof. First assume that β is a non-simple beta-number. By Corollary 5.4, β is not emergent, so the result follows from Theorem H′.

Now assume that β is a simple beta-number. By Corollary 5.4, β is emergent. Theorem 5.22 gives that $C^{0,\alpha}(I)$ is equal to the union of $\operatorname{Crit}^\alpha(\beta)$ and the closure of $\operatorname{Lock}^\alpha(U_\beta)$, so it suffices to show that $\operatorname{Crit}^\alpha(\beta)$ is a subset of the closure of $\operatorname{Lock}^\alpha(U_\beta)$ (as $\operatorname{Lock}^\alpha(U_\beta)$ is by definition an open subset of $C^{0,\alpha}(I)$). If $\phi \in \operatorname{Crit}^\alpha(\beta)$ then from (5.2) we see that the periodic measure supported by $\mathcal{O}^*_\beta(1)$ is (U_β, ϕ)-maximizing, so $\phi \in \mathscr{P}^\alpha(U_\beta)$, and therefore ϕ belongs to the closure of $\operatorname{Lock}^\alpha(U_\beta)$ by Theorem 5.20. Theorem G′ follows. □

6. Joint TPO: beta-transformations

In this section, we prove Theorem C, the Joint TPO theorem for beta-transformations, and then deduce Theorem F (Individual TPO for generic potentials). In fact Theorem C will follow from a stronger Theorem C′, that in particular establishes the joint typical periodic optimization property for both beta-transformations and upper beta-transformations. The proof of Theorem C′ comprises two steps. The first step, consisting of the key joint perturbation result (Theorem 6.4), is to prove that for any *non-simple beta-number* β, any $\phi \in \operatorname{Lock}^\alpha(U_\beta)$ that is uniquely U_β-maximized on a periodic orbit $\mathcal{O}_\beta$, and any $\gamma < \beta$ sufficiently close to β, we can perform a small perturbation of ϕ so as to render it uniquely U_γ-maximized by the U_γ-periodic orbit $(h_\gamma \circ \pi_\beta)(\mathcal{O}_\beta)$. In this first step, the beta-transformations Mañé lemma (Theorem 4.9) is key to making the perturbation, and the subsequent analysis is inspired by ideas in [Boc19, Section 4] (itself based on the preprint version of [HLMXZ19]), together with some more careful estimates exploiting various specific characteristics of the family of beta-transformations. The choice of β as a non-simple beta-number is an essential feature of this step, since on the one hand the functions $u^\pm_{\beta,\phi}$ in the Mañé lemma (Theorem 4.9) are required to be piecewise α-Hölder (which is the case if β is a beta-number, cf. Remark 4.10), and on the other hand the perturbative analysis (specifically, a shadowing result, Corollary 6.2) requires that the orbit $\mathcal{O}_\beta$ does not contain the point 1, thereby forcing β to be non-simple. The second step consists of combining the result from the first step with the fact that non-simple beta-numbers are dense in $(1, +\infty)$, and invoking Theorem G′ (the strong version of Individual TPO for beta-numbers), in order to deduce Theorem C′.

6.1. **Shadowing for beta-transformations.** As a preliminary step, we first give a simple bound on the distance between $h_\beta(\underline{a})$ and $h_\gamma(\underline{a})$ for $\beta > \gamma > 1$ and $\underline{a} \in X_\gamma$.

Lemma 6.1. *If* $1 < \gamma < \beta$ *then for each* $\underline{a} \in X_\gamma$,

$$|h_\gamma(\underline{a}) - h_\beta(\underline{a})| \leqslant \frac{(\beta-\gamma)\gamma^2}{\beta(\gamma-1)^2}. \tag{6.1}$$

Proof. Writing $\underline{a} = a_1 a_2 \dots$, since $\underline{a} \in X_\gamma$ then $a_n \leqslant \gamma$ for each $n \in \mathbb{N}$, so from (3.7),

$$|h_\gamma(\underline{a}) - h_\beta(\underline{a})| = \sum_{n=1}^{+\infty} a_n \frac{\beta^n - \gamma^n}{\beta^n\gamma^n} \leqslant \sum_{n=1}^{+\infty} \frac{\beta^n - \gamma^n}{\beta^n\gamma^{n-1}},$$

and therefore

$$|h_\gamma(\underline{a}) - h_\beta(\underline{a})| \leqslant (\beta-\gamma)\sum_{n=1}^{+\infty} \frac{\sum_{i=0}^{n-1}\beta^i\gamma^{n-1-i}}{\beta^n\gamma^{n-1}} \leqslant (\beta-\gamma)\sum_{n=1}^{+\infty}\frac{n}{\beta\gamma^{n-1}} = \frac{(\beta-\gamma)\gamma^2}{\beta(\gamma-1)^2},$$

as required. □

From Lemma 6.1, we are able to estimate the distance between a U_β-orbit $\mathcal{O}_\beta$ and its image $(h_\gamma \circ \pi_\beta)(\mathcal{O}_\beta)$; this rather explicit version of a shadowing lemma will simplify our subsequent proof of Theorem 6.4, where it is used systematically.

Corollary 6.2. *Fix $\beta > 1$, and let $\mathcal{O}_\beta \in \mathrm{Per}(U_\beta)$ be such that $1 \notin \mathcal{O}_\beta$. Then there exists $c \in (0, \beta-1)$ such that if $\gamma \in (\beta - c, \beta)$, then $h_\gamma \circ \pi_\beta$ is well defined on $\mathcal{O}_\beta$, and:*

(i) *$\mathcal{O}_\gamma := (h_\gamma \circ \pi_\beta)(\mathcal{O}_\beta)$ is a U_γ-periodic orbit with* $\mathrm{card}\,\mathcal{O}_\gamma = \mathrm{card}\,\mathcal{O}_\beta$.

(ii) *There exists $M > 0$ such that for each $x \in \mathcal{O}_\beta$,*

$$|(h_\gamma \circ \pi_\beta)(x) - x| \leqslant M(\beta - \gamma). \tag{6.2}$$

(iii) *If $\mathcal{O}_\beta \neq \{0\}$ then there exists $s > 0$ such that for each $x \in \mathcal{O}_\beta$,*

$$s(\beta - \gamma) \leqslant |(h_\gamma \circ \pi_\beta)(x) - x|. \tag{6.3}$$

Proof. Since $1 \notin \mathcal{O}_\beta$, Proposition 3.12 (ii) implies that $\mathcal{O}_\beta$ is also T_β-periodic. Applying Lemma 3.19 with $\mathcal{K} = \mathcal{O}_\beta$, there exists $c > 0$ such that $\mathcal{O}_\beta \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta - c, \beta)$, and without loss of generality c may be chosen to be strictly smaller than $\beta - 1$. For each $\gamma \in (\beta - c, \beta)$, Lemma 3.20 implies that $\pi_\beta\big(H_\beta^\gamma\big) \subseteq \mathcal{S}_\gamma \subseteq X_\gamma$, and since h_γ is defined on X_γ (see (3.7)), it follows that $h_\gamma \circ \pi_\beta$ is well defined on H_β^γ, hence in particular on $\mathcal{O}_\beta$, as required.

(i) By Proposition 3.9 (iii) and (vi), $\pi_\beta(\mathcal{O}_\beta)$ is a σ-periodic orbit with $\mathrm{card}\,\pi_\beta(\mathcal{O}_\beta) = \mathrm{card}\,\mathcal{O}_\beta$. So by Proposition 3.27 (iv), $\mathcal{O}_\gamma = (h_\gamma \circ \pi_\beta)(\mathcal{O}_\beta)$ is a U_γ-periodic orbit with $\mathrm{card}\,\mathcal{O}_\gamma = \mathrm{card}\,\pi_\beta(\mathcal{O}_\beta) = \mathrm{card}\,\mathcal{O}_\beta$.

(ii) Now $c < \beta - 1$, so for $\gamma \in (\beta - c, \beta)$ the expression $\frac{\gamma^2}{\beta(\gamma-1)^2}$ has finite upper bound M (equal to $\frac{(\beta-c)^2}{\beta(\beta-c-1)^2}$). Moreover, applying Proposition 3.9 (iv) and Lemma 6.1, for each $x \in \mathcal{O}_\beta$, we have

$$|(h_\gamma \circ \pi_\beta)(x) - x| = |(h_\gamma \circ \pi_\beta)(x) - (h_\beta \circ \pi_\beta)(x)| \leqslant (\beta - \gamma)\gamma^2\beta^{-1}(\gamma-1)^{-2} \leqslant M(\beta - \gamma). \tag{6.4}$$

(iii) Since $\mathcal{O}_\beta \neq \{0\}$, then $(0)^\infty \notin \pi_\beta(\mathcal{O}_\beta)$. Since $\pi_\beta(\mathcal{O}_\beta)$ is finite, there exists $N \in \mathbb{N}$ such that if $x \in \mathcal{O}_\beta$ then $(0)^N 1(0)^\infty \prec \pi_\beta(x)$. Let $x \in \mathcal{O}_\beta$, and write $\pi_\beta(x) = a_1 a_2 \dots$, and let $n_x \leqslant N + 1$ be the smallest integer such that $a_{n_x} \neq 0$, so that by (6.4) and (3.7),

$$|(h_\gamma \circ \pi_\beta)(x) - x| = |(h_\gamma \circ \pi_\beta)(x) - (h_\beta \circ \pi_\beta)(x)| = \sum_{i=n_x}^{+\infty} a_i \frac{\beta^i - \gamma^i}{\beta^i \gamma^i} \geqslant \frac{\beta^{n_x} - \gamma^{n_x}}{\beta^{n_x}\gamma^{n_x}} \geqslant \frac{\beta - \gamma}{\beta^{n_x}\gamma} \geqslant \frac{\beta - \gamma}{\beta^{N+2}}.$$

Thus (6.3) holds with $s := \beta^{-N-2}$. □

We will need the following expression for the ergodic supremum (the analogue for continuous maps is well known, see e.g. [Je19, Proposition 2.2]):

Lemma 6.3. *Given any $\beta > 1$ and any $\phi \in C(I)$,*

$$Q(U_\beta, \phi) = \sup_{x \in I} \liminf_{n \to +\infty} \frac{1}{n} S_n^{U_\beta} \phi(x).$$

Proof. Since $\sigma \colon X_\beta \to X_\beta$ is a continuous map on a compact metric space, and $\phi \circ h_\beta$ is continuous, [Je19, Proposition 2.2] gives

$$Q\big(\sigma|_{X_\beta}, \phi \circ h_\beta\big) = \sup_{\underline{a} \in X_\beta} \liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma (\phi \circ h_\beta)(\underline{a}). \tag{6.5}$$

Applying Lemma 3.26 (ii) with $W = I$ gives $X_\beta = \pi_\beta(Z_\beta) \cup \pi_\beta^*(I)$. By Proposition 3.9 (ii) and (3.15), for each $\underline{b} \in \pi_\beta(Z_\beta)$ there exists $m \in \mathbb{N}$ such that $\sigma^m(\underline{b}) = (0)^\infty$. Thus, noting that $(0)^\infty = \pi_\beta^*(0)$, we obtain that

$$\liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)(\underline{b}) = \liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)((0)^\infty) \leqslant \sup_{\underline{a} \in \pi_\beta^*(I)} \liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)(\underline{a})$$

for each $\underline{b} \in \pi_\beta(Z_\beta)$. This, together with the fact that $X_\beta = \pi_\beta(Z_\beta) \cup \pi_\beta^*(I)$, implies that

$$\sup_{\underline{a} \in X_\beta} \liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)(\underline{a}) = \sup_{\underline{a} \in \pi_\beta^*(I)} \liminf_{n \to +\infty} \frac{1}{n} S_n^\sigma(\phi \circ h_\beta)(\underline{a}). \tag{6.6}$$

By Proposition 3.9 (iv) and (v),

$$\sup_{x\in I}\liminf_{n\to+\infty}\frac{1}{n}S_n^{U_\beta}\phi(x)=\sup_{x\in I}\liminf_{n\to+\infty}\frac{1}{n}S_n^{\sigma}(\phi\circ h_\beta)\big(\pi_\beta^*(x)\big)=\sup_{\underline{a}\in\pi_\beta^*(I)}\liminf_{n\to+\infty}\frac{1}{n}S_n^{\sigma}(\phi\circ h_\beta)(\underline{a}). \tag{6.7}$$

So combining (6.5), (6.6), (6.7), and using Proposition 3.27 (iii), completes the proof. □

6.2. **Joint perturbation for beta-transformations.** The following key Theorem 6.4, valid for *non-simple beta-numbers*, is the first perturbative step (described at the start of this section) towards proving Theorem C′. As noted previously, it has a similar character to the joint perturbation theorems for expanding maps (Theorem 2.7) and for Anosov diffeomorphisms (Theorems 2.12 and 2.13), though here the analysis is more intricate, and the proof is lengthier.

Theorem 6.4 (Joint Perturbation: beta-transformations). *Fix $\alpha\in(0,1]$, and suppose $\beta>1$ is a non-simple beta-number.*

(i) *If $\mathcal{O}_\beta$ is a T_β-periodic orbit, then there exist C_1, $C_2>0$ such that if $\gamma\in(\beta-C_2,\beta)$ and $\phi\in C^{0,\alpha}(I)$ with $\mathcal{M}_{\max}(T_\beta,\phi)=\{\mu_{\mathcal{O}_\beta}\}$, then the T_γ-periodic measure $\mu_{\mathcal{O}_\gamma}$ supported by $\mathcal{O}_\gamma:=(h_\gamma\circ\pi_\beta)(\mathcal{O}_\beta)$, is the unique T_γ-maximizing measure for the function*

$$\phi-2C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}d(\cdot,\mathcal{O}_\gamma)^\alpha. \tag{6.8}$$

(ii) *If $\mathcal{O}_\beta$ is a U_β-periodic orbit, then there exist C_1, $C_2>0$ such that if $\gamma\in(\beta-C_2,\beta)$ and $\phi\in C^{0,\alpha}(I)$ with $\mathcal{M}_{\max}(U_\beta,\phi)=\{\mu_{\mathcal{O}_\beta}\}$, then the U_γ-periodic measure $\mu_{\mathcal{O}_\gamma}$ supported by $\mathcal{O}_\gamma:=(h_\gamma\circ\pi_\beta)(\mathcal{O}_\beta)$, is the unique U_γ-maximizing measure for the function defined by (6.8).*

Proof. Fix a non-simple beta-number $\beta>1$.

(i) First we show that part (i) follows readily from part (ii). Given a T_β-periodic orbit $\mathcal{O}_\beta$, Proposition 3.12 (ii) implies that $\mathcal{O}_\beta$ is also U_β-periodic. Assuming that part (ii) of this theorem has been proved, let C_1, $C_2>0$ be as in (ii), let $\gamma\in(\beta-C_2,\beta)$ and $\gamma'\in(\beta-C_2,\gamma)$, and recall that $\mathcal{O}_\gamma=(h_\gamma\circ\pi_\beta)(\mathcal{O}_\beta)$. By the definition of $h_\beta\colon X_\beta\to I$ (see (3.7)), $\max\{x:x\in\mathcal{O}_\gamma\}<\max\{x:x\in\mathcal{O}_{\gamma'}\}\leqslant 1$, so $1\notin\mathcal{O}_\gamma$. By (ii) and Proposition 3.12 (ii), $\mathcal{O}_\gamma$ is also a T_γ-periodic orbit, so $\mu_{\mathcal{O}_\gamma}\in\mathcal{M}(I,T_\gamma)$. By (ii) we know that $\mu_{\mathcal{O}_\gamma}$ is the unique U_γ-maximizing measure for the function given by (6.8), so the fact that $\mathcal{M}(I,T_\gamma)\subseteq\mathcal{M}(I,U_\gamma)$ (cf. Proposition 3.12 (iii)) implies that $\mu_{\mathcal{O}_\gamma}$ is also the unique T_γ-maximizing measure for this function, as required.

(ii) To prove (ii), suppose that $\mathcal{O}_\beta$ is a U_β-periodic orbit. Since β is non-simple, the point 1 is not U_β-periodic, and therefore not an element of $\mathcal{O}_\beta$. By Corollary 6.2 (ii) and (iii), it follows that there exist constants $M>0$ and $c\in(0,\beta-1)$ such that if $\gamma\in(\beta-c,\beta)$ then (6.2) holds, and if moreover $\mathcal{O}_\beta\neq\{0\}$ then there also exists $s>0$ such that (6.3) holds. Let us write $u:=u^-_{\beta,\phi}$ for the bounded left-continuous (cf. Theorem 4.9 (i)) function defined by (4.43), and write $K_\beta:=K_{\alpha,\beta}=\frac{1}{\beta^\alpha-1}$ (cf. (4.3)). In the following proof we shall systematically exploit two inequalities. The first is that

$$\psi_\beta:=\overline{\phi}+u-u\circ U_\beta\leqslant 0, \tag{6.9}$$

where $\overline{\phi}:=\phi-Q(U_\beta,\phi)$, which is a consequence of (4.45) and Theorem 4.9 (ii). The second is that

$$u(x)-u(y)\leqslant K_\beta|\phi|_\alpha|x-y|^\alpha \tag{6.10}$$

holds if $x<y$ and $[x,y)\cap\mathcal{O}^*_\beta(1)=\emptyset$, and also holds if $y<x$. Note that if $x<y$ and $[x,y)\cap\mathcal{O}^*_\beta(1)=\emptyset$ then (6.10) follows from Theorem 4.9 (iii) and the left continuity of u, whereas if $y<x$ then (6.10) follows from Lemma 4.5, together with (4.11) and (4.43).

Recall (cf. Subsection 1.5) that if $F\subseteq I$ is a finite set, then $\Delta(F)$ denotes its minimum interpoint distance, i.e., $\Delta(F)=\min\{|x-y|:x,\,y\in F,\,x\neq y\}$ if $\operatorname{card}F\geqslant 2$ and $\Delta(F)=+\infty$ if $\operatorname{card}F=1$. In the following, the finite set F will be chosen as either a periodic orbit or a preperiodic orbit.

We define the following notation for various constants[27] that will be used in our perturbative arguments (recall that D_β is the discontinuity set, defined in (3.5)):

$$p := \operatorname{card} \mathcal{O}_\beta, \tag{6.11}$$

$$r := \begin{cases} \min\left\{\frac{1}{3}d(\mathcal{O}_\beta, D_\beta),\ \frac{1}{4}\Delta(\mathcal{O}_\beta),\ \frac{1}{2}d\big(\mathcal{O}_\beta, \mathcal{O}^*_\beta(1)\big)\right\} & \text{if } \mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset, \\ \min\left\{\frac{1}{3}d(\mathcal{O}_\beta, D_\beta),\ \frac{1}{4}\Delta(\mathcal{O}_\beta)\right\} & \text{if } \mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) \neq \emptyset, \end{cases} \tag{6.12}$$

$$C_2 := \min\left\{c,\ \frac{1}{2},\ \frac{1}{2}\Delta\big(\mathcal{O}^*_\beta(1)\big),\ \frac{1}{2M}\Delta\big(\mathcal{O}^*_\beta(1)\big),\ M^{-1}r,\ \beta r\right\}, \tag{6.13}$$

$$L_1 := 1 + \frac{1}{(\beta - C_2)^\alpha - 1} + 2\beta^\alpha K_\beta, \tag{6.14}$$

$$L_2 := 3K_\beta + 1 + M^\alpha, \tag{6.15}$$

$$C_1 := \max\left\{1,\ L_2 \cdot C_2^{\alpha/2}\beta^\alpha r^{-\alpha},\ r^{-\alpha}\beta^\alpha(p + 1 + L_1)L_2\right\}. \tag{6.16}$$

Note that $C_2 \leqslant c < \beta - 1$, so that $\beta - C_2 > 1$ and $\ln(\beta - C_2) > 0$. If $\mathcal{O}_\beta \neq \{0\}$, we define the additional constants

$$L_3 := 1 + \frac{1}{\ln(\beta - C_2)}\left(\ln\beta + \frac{1}{\alpha}\ln L_2 - \ln s\right), \tag{6.17}$$

$$L_4 := \frac{s^\alpha}{\beta^\alpha}L_1 + pL_2 + L_3L_2 + \frac{L_2}{e\alpha\ln(\beta - C_2)}, \tag{6.18}$$

and make the additional assumption that $C_1 \geqslant r^{-\alpha}\beta^\alpha(L_4 + L_2)$, in other words, if $\mathcal{O}_\beta \neq \{0\}$ we re-define C_1 by

$$C_1 := \max\left\{1,\ L_2 \cdot C_2^{\alpha/2}\beta^\alpha r^{-\alpha},\ r^{-\alpha}\beta^\alpha(p + 1 + L_1)L_2,\ r^{-\alpha}\beta^\alpha(L_4 + L_2)\right\}. \tag{6.19}$$

Note that the constants L_1, L_2, L_3, and L_4 are defined purely for convenience, in order to lighten the notation.

Now suppose $\gamma \in (\beta - C_2, \beta)$, and define

$$r_\gamma := \begin{cases} \min\left\{d(\mathcal{O}_\gamma, D_\gamma),\ \frac{1}{2}\Delta(\mathcal{O}_\gamma),\ d\big(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\big)\right\} & \text{if } \mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset, \\ \min\left\{d(\mathcal{O}_\gamma, D_\gamma),\ \frac{1}{2}\Delta(\mathcal{O}_\gamma)\right\} & \text{if } \mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) \neq \emptyset, \end{cases} \tag{6.20}$$

$$\psi_\gamma := \overline{\phi} + u - u \circ U_\gamma, \tag{6.21}$$

$$\tau := (3K_\beta + 1)|\phi|_\alpha(\beta - \gamma)^\alpha. \tag{6.22}$$

Before embarking on the perturbative part of the proof, we shall establish the following three preparatory claims:

Claim 1. $r > 0$, $C_2 > 0$, and $r_\gamma \geqslant r$.

Proof of Claim 1. First we show that $r > 0$. If there were some point $x \in \mathcal{O}_\beta \cap D_\beta$, then $U_\beta(x)$ would belong to $\mathcal{O}_\beta \cap \{1\}$ by (3.5), but this contradicts the fact that 1 is not a U_β-periodic point, since β is a non-simple beta-number; thus in fact $d(\mathcal{O}_\beta, D_\beta) > 0$. The strict positivity of the minimum interpoint distance $\Delta(\mathcal{O}_\beta)$ follows directly from its definition, and the strict positivity of $d\big(\mathcal{O}_\beta, \mathcal{O}^*_\beta(1)\big)$ clearly does hold in the case where $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset$. Thus we have shown that $r > 0$. It readily follows that $C_2 > 0$ as well (since the terms not involving r on the right-hand side of (6.13) are clearly positive).

[27]Note that the form of the presentation of C_2 in (6.13) is for ease of reference in subsequent calculations rather than for economy of notation, as certain elements of the set on the right-hand side are manifestly dominated by others. For future reference, we note that the bound $C_2 \leqslant c$ is needed so as to invoke Corollary 6.2, while $C_2 \leqslant \frac{1}{2}$ and $C_2 \leqslant \frac{1}{2}\Delta(\mathcal{O}^*_\beta(1))$ are used in proving Claim 3, the inequalities $C_2 \leqslant M^{-1}r$ and $C_2 \leqslant \beta r$ are used in proving Claim 1, and $C_2 \leqslant \frac{1}{2M}\Delta(\mathcal{O}^*_\beta(1))$ is used in proving that $d(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)) \geqslant s(\beta - \gamma)$ in Subcase (ii) of Case C. As for the constant C_1, the bound $C_1 \geqslant 1$ is assumed in order to simplify calculations, the second term in (6.16) is needed to check that $\rho \leqslant r/\beta$ (cf. (6.42)), the third term is used in deriving (6.53), and the fourth term in (6.19) is used in deriving (6.72).

It remains to show that $r_\gamma \geqslant r$, and for this we will use the definitions of the constants r, r_γ, C_2, and that of the discontinuity set D_β (see (3.5)). Consider $x \in \mathcal{O}_\gamma$ and $x' \in \mathcal{O}_\beta$ with $x = (h_\gamma \circ \pi_\beta)(x')$. Suppose $z = i/\gamma \in D_\gamma$ for some non-negative integer $i \leqslant \gamma$, and fix $w \in \mathcal{O}^*_\beta(1)$. By Corollary 6.2, and using that $C_2 \leqslant M^{-1}r$, we have $|x - x'| \leqslant M(\beta - \gamma) \leqslant MC_2 \leqslant r$. Now set $z' \coloneqq i/\beta \in D_\beta$, and using that $C_2 \leqslant \beta r$, we have $|z - z'| = \frac{i(\beta-\gamma)}{\beta\gamma} \leqslant \frac{C_2}{\beta} \leqslant r$. Hence, if $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) \neq \emptyset$, the above inequalities, together with the triangle inequality, and the definition of r, yield

$$|x - z| \geqslant |x' - z'| - |x' - x| - |z' - z| \geqslant d(\mathcal{O}_\beta, D_\beta) - r - r \geqslant 3r - 2r = r. \tag{6.23}$$

If $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset$, then (6.23) also holds, and the definition of r in this case additionally yields

$$|x - w| \geqslant |x' - w| - |x - x'| \geqslant d\big(\mathcal{O}_\beta, \mathcal{O}^*_\beta(1)\big) - r \geqslant 2r - r = r. \tag{6.24}$$

From (6.23) we deduce that $d(\mathcal{O}_\gamma, D_\gamma) \geqslant r$, and if $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset$ then from (6.24) we deduce that $d\big(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\big) \geqslant r$. If $p = 1$, by Corollary 6.2 (i), $\operatorname{card} \mathcal{O}_\gamma = 1$, so $\Delta(\mathcal{O}_\gamma) = +\infty$. As $d(\mathcal{O}_\gamma, D_\gamma) \geqslant r$ and if $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset$ then $d\big(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\big) \geqslant r$, we obtain $r_\gamma \geqslant r$. If $p \geqslant 2$, by Corollary 6.2 (i), $\operatorname{card} \mathcal{O}_\gamma \geqslant 2$. In this case, consider $y \in \mathcal{O}_\gamma$ and $y' \in \mathcal{O}_\beta$ with $x \neq y$ and $y = (h_\gamma \circ \pi_\beta)(y')$. Similarly, we have $|y - y'| \leqslant r$. Recalling that $|x - x'| \leqslant r$, we have

$$|x - y| \geqslant |x' - y'| - |x' - x| - |y' - y| \geqslant \Delta(\mathcal{O}_\beta) - 2r \geqslant 4r - 2r = 2r, \tag{6.25}$$

which implies that $\frac{1}{2}\Delta(\mathcal{O}_\gamma) \geqslant r$. Recalling that $d(\mathcal{O}_\gamma, D_\gamma) \geqslant r$ and if $\mathcal{O}_\beta \cap \mathcal{O}^*_\beta(1) = \emptyset$ then $d\big(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\big) \geqslant r$, we obtain that $r_\gamma \geqslant r$. Therefore, Claim 1 is proved.

Claim 2. For each $x \in \mathcal{O}_\gamma$, if $s, t \in B(x, r)$ then $U_\gamma(s) - U_\gamma(t) = \gamma(s - t)$, moreover, $d(s, \mathcal{O}_\gamma) = |s - x|$.

Proof of Claim 2. Note that $r \leqslant r_\gamma$ by Claim 1. Given $x \in \mathcal{O}_\gamma$ and $s, t \in B(x, r) \subseteq B(x, r_\gamma)$, note first that since $r_\gamma \leqslant d(\mathcal{O}_\gamma, D_\gamma)$, then $B(x, r_\gamma) \cap D_\gamma = \emptyset$, thus $U_\gamma(s) - U_\gamma(t) = \gamma(s - t)$. Secondly, since $|s - x| < r_\gamma \leqslant \frac{1}{2}\Delta(\mathcal{O}_\gamma)$, then x is the closest point in $\mathcal{O}_\gamma$ to s, so that $d(s, \mathcal{O}_\gamma) = |s - x|$. So Claim 2 is proved.

The following[28] Claim 3 means that the function u can be regarded as a sub-action for (U_γ, ϕ):

Claim 3. $\psi_\gamma \leqslant \tau$.

Proof of Claim 3. Recall that $\lfloor x \rfloor' \coloneqq \max\{n \in \mathbb{Z} : n < x\}$ is a non-decreasing function that only takes integer values.

Fix $x \in I$. First consider the case where $\lfloor \gamma x \rfloor' = \lfloor \beta x \rfloor'$. By the definition of U_β (see (3.2)), $U_\beta(x) = U_\gamma(x) + x(\beta - \gamma) > U_\gamma(x)$, so by (6.21), (6.9), and (6.10),

$$\psi_\gamma(x) = \psi_\beta(x) + u(U_\beta(x)) - u(U_\gamma(x)) \leqslant K_\beta |\phi|_\alpha (U_\beta(x) - U_\gamma(x))^\alpha \leqslant K_\beta |\phi|_\alpha (\beta - \gamma)^\alpha. \tag{6.26}$$

Next consider the case where $\lfloor \gamma x \rfloor' < \lfloor \beta x \rfloor'$, and set $y \coloneqq x - \frac{\beta - \gamma}{\beta}$. Since $\beta - \gamma < C_2 \leqslant \frac{1}{2}$,

$$\gamma y = \gamma x - \gamma\beta^{-1}(\beta - \gamma) = \beta x - \big(x + \gamma\beta^{-1}\big)(\beta - \gamma) > \beta x - 2(\beta - \gamma) > \beta x - 1. \tag{6.27}$$

Thus, $\lfloor \beta x \rfloor' - 1 \leqslant \lfloor \gamma y \rfloor' \leqslant \lfloor \gamma x \rfloor' < \lfloor \beta x \rfloor'$. So $\lfloor \gamma y \rfloor' = \lfloor \gamma x \rfloor' = \lfloor \beta x \rfloor' - 1$. On the other hand, we have $\gamma y < \beta y = \beta x - (\beta - \gamma) = \gamma x + (x - 1)(\beta - \gamma) \leqslant \gamma x$, thus

$$\lfloor \gamma y \rfloor' = \lfloor \beta y \rfloor'. \tag{6.28}$$

In view of (6.28), the inequality (6.26) holds (with y replacing x), in other words,

$$\psi_\gamma(y) \leqslant K_\beta |\phi|_\alpha (\beta - \gamma)^\alpha. \tag{6.29}$$

[28] Note that Claim 3 represents an analogue of the inequality (2.25), established in the context of expanding maps. While (2.25) is obtained relatively easily, the proof of Claim 3 is more delicate (in particular, it need not hold if either $\gamma > \beta$ or β is not a beta-number), and exploits particular properties of beta-transformations.

Now $y < x$, so by (6.10) we have $u(x) - u(y) \leqslant K_\beta |\phi|_\alpha |x-y|^\alpha$, and combining this with (6.29) and (6.21) gives

$$\begin{aligned}\psi_\gamma(x) - |u(U_\gamma(x)) - u(U_\gamma(y))| &\leqslant \psi_\gamma(y) + |\phi(x) - \phi(y)| + u(x) - u(y) \\ &\leqslant (2K_\beta + 1)|\phi|_\alpha (x-y)^\alpha \leqslant (2K_\beta+1)|\phi|_\alpha(\beta-\gamma)^\alpha.\end{aligned} \tag{6.30}$$

The last inequality above follows from $x - y = (\beta-\gamma)/\beta < \beta - \gamma$. Since $C_2 \leqslant \frac{1}{2}\Delta(\mathcal{O}^*_\beta(1))$, and $1 \in \mathcal{O}^*_\beta(1)$, we have $\mathcal{O}^*_\beta(1) \cap (1-2C_2, 1) = \emptyset$. By (6.27) and $\lfloor \beta x\rfloor' = \lfloor \gamma y \rfloor' + 1$, we have $\gamma y > \beta x - 2C_2 \geqslant \lfloor \gamma y\rfloor' + 1 - 2C_2$, which implies that $U_\gamma(y) \in (1-2C_2, 1]$. Since $\lfloor \gamma y\rfloor' = \lfloor \gamma x\rfloor'$, we have $1 - 2C_2 < U_\gamma(y) < U_\gamma(y) + \gamma(x-y) = U_\gamma(x) \leqslant 1$. Thus, applying (6.10) (with the points x and y replaced by $U_\gamma(x)$ and $U_\gamma(y)$) gives

$$|u(U_\gamma(x)) - u(U_\gamma(y))| \leqslant K_\beta|\phi|_\alpha(\gamma(x-y))^\alpha < K_\beta |\phi|_\alpha (\beta-\gamma)^\alpha, \tag{6.31}$$

where the last inequality follows from $x - y = (\beta-\gamma)/\beta < (\beta-\gamma)/\gamma$. Combining (6.26), (6.30), (6.31), and recalling that $\tau = (3K_\beta+1)|\phi|_\alpha(\beta-\gamma)^\alpha$ (cf. (6.22)), we obtain that $\psi_\gamma \leqslant \tau$, so Claim 3 is proved.

Having proved the above Claims 1, 2, and 3, we are now ready to begin the perturbative part of the proof, by defining the functions

$$\phi'_\gamma := \overline{\phi} - C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha \quad \text{and} \tag{6.32}$$

$$\psi'_\gamma := \psi_\gamma - C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha = \phi'_\gamma + u - u\circ U_\gamma, \tag{6.33}$$

noting that the second equality in (6.33) follows from (6.21) and (6.32).

Note that the function defined (cf. (6.8)) in the statement of the theorem differs from ϕ'_γ by $Q(U_\beta,\phi) - C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}d(\cdot,\mathcal{O}_\gamma)^\alpha$, and this latter function clearly has $\mu_{\mathcal{O}_\gamma}$ as its unique U_γ-maximizing measure, since the maximum of the function is attained uniquely on the U_γ-periodic orbit $\mathcal{O}_\gamma$. Consequently, if it can be shown that the periodic measure $\mu_{\mathcal{O}_\gamma}$ is (U_γ, ϕ'_γ)-maximizing, then it will follow that this measure is the *unique* U_γ-maximizing measure for the function defined in (6.8), and the theorem will be proved.

We therefore aim to prove that $\mu_{\mathcal{O}_\gamma}$ is (U_γ,ϕ'_γ)-maximizing. Using (6.32), combined with Corollary 6.2 and the assumption that the unique (U_β,ϕ)-maximizing measure is supported by the period-p orbit $\mathcal{O}_\beta$, we see that

$$\eta := \int_I \phi'_\gamma \,\mathrm{d}\mu_{\mathcal{O}_\gamma} = \int_I \overline{\phi}\,\mathrm{d}\mu_{\mathcal{O}_\gamma} \geqslant \int_I \overline{\phi}\,\mathrm{d}\mu_{\mathcal{O}_\beta} - \frac{1}{p}\sum_{x\in\mathcal{O}_\beta}\bigl|\bigl(\overline{\phi}\circ h_\gamma \circ \pi_\beta\bigr)(x) - \overline{\phi}(x)\bigr| \geqslant -|\phi|_\alpha M^\alpha(\beta-\gamma)^\alpha, \tag{6.34}$$

and deduce from (6.34), using (6.15) and (6.22), that

$$\tau - \eta \leqslant L_2|\phi|_\alpha(\beta-\gamma)^\alpha. \tag{6.35}$$

Since $\psi_\gamma \leqslant \tau$ by Claim 3, and $\int \psi_\gamma\,\mathrm{d}\mu_{\mathcal{O}_\gamma} = \int\overline{\phi}\,\mathrm{d}\mu_{\mathcal{O}_\gamma} = \int \phi'_\gamma\,\mathrm{d}\mu_{\mathcal{O}_\gamma} = \eta$ (cf. (6.21) and (6.34)), then

$$\tau - \eta \geqslant 0. \tag{6.36}$$

Now $\mathcal{O}_\gamma$ is a U_γ-periodic orbit, by Corollary 6.2 (i), so $\mu_{\mathcal{O}_\gamma} \in \mathcal{M}(I, U_\gamma)$, and $\eta = \int_I \phi'_\gamma\,\mathrm{d}\mu_{\mathcal{O}_\gamma}$ (cf. (6.34)), so to show that $\mu_{\mathcal{O}_\gamma}$ is a (U_γ,ϕ'_γ)-maximizing measure, it suffices to prove that $Q(U_\gamma,\phi'_\gamma) \leqslant \eta$. But $Q(U_\gamma,\phi'_\gamma) = \sup_{x\in I}\liminf_{n\to+\infty}\frac{1}{n}S_n^{U_\gamma}\phi'_\gamma(x)$ by Lemma 6.3, so in view of (6.33), and the fact that u is bounded, if we can show that

$$\liminf_{n\to+\infty}\frac{1}{n}S_n^{U_\gamma}\psi'_\gamma(x) \leqslant \eta \quad \text{for all } x \in I, \tag{6.37}$$

then the required fact that $\mu_{\mathcal{O}_\gamma}$ is (U_γ,ϕ'_γ)-maximizing will follow.

We therefore aim to prove (6.37). Defining

$$\rho := (\tau-\eta)^{1/\alpha}(C_1|\phi|_\alpha)^{-1/\alpha}(\beta-\gamma)^{-1/2}, \tag{6.38}$$

note that the definition of ψ'_γ (cf. (6.33)), together with $\psi_\gamma \leqslant \tau$ (by Claim 3), implies that

$$\psi'_\gamma(x) \leqslant \eta \quad \text{for all } x \notin B(\mathcal{O}_\gamma, \rho). \tag{6.39}$$

Now fix an arbitrary point $x \in I$. In order to show that $\liminf_{n\to+\infty} \frac{1}{n} S_n^{U_\gamma} \psi'_\gamma(x) \leqslant \eta$ we will recursively construct a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathcal{O}^*_\gamma(x)$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ satisfying

$$x_{k+1} = U_\gamma^{n_k}(x_k) \quad \text{and} \quad S_{n_k}^{U_\gamma} \psi'_\gamma(x_k) \leqslant n_k \eta.$$

Base step. Define $x_1 \coloneqq x$.

Recursive step. Assume that for some $t \in \mathbb{N}$, $\{x_k\}_{k=1}^t$ and $\{n_k\}_{k=1}^{t-1}$ are defined. We now divide our discussion into three cases, the third of which requires some delicate analysis.

Case A. Assume $x_t \in \mathcal{O}_\gamma$. Then define $n_t \coloneqq p$ and $x_{t+1} \coloneqq U_\gamma^{n_t}(x_t) = x_t$. Thus, by (6.33) and (6.34), we have

$$S_{n_t}^{U_\gamma} \psi'_\gamma(x_t) = n_t \eta. \tag{6.40}$$

Case B. Assume $x_t \notin B(\mathcal{O}_\gamma, \rho)$. Then define $n_t \coloneqq 1$ and $x_{t+1} \coloneqq U_\gamma^{n_t}(x_t) = U_\gamma(x_t)$, so that (6.39) gives

$$S_{n_t}^{U_\gamma} \psi'_\gamma(x_t) = \psi'_\gamma(x_t) \leqslant \eta. \tag{6.41}$$

Case C. Assume $x_t \in B(\mathcal{O}_\gamma, \rho) \smallsetminus \mathcal{O}_\gamma$. Note that $C_1 \geqslant \frac{L_2 \cdot C_2^{\alpha/2} \beta^\alpha}{r^\alpha}$ by (6.16), and $\beta - \gamma < C_2$, so using (6.35) and (6.38) it follows that

$$\rho \leqslant r/\beta. \tag{6.42}$$

Let y be a point in $\mathcal{O}_\gamma$ that is closest to x_t, in the sense that

$$|x_t - y| = d(x_t, \mathcal{O}_\gamma), \tag{6.43}$$

so $x_t \in B(y, \rho) \subseteq B(y, r)$ by the Case C assumption together with (6.42). Next, define

$$N \coloneqq \min\bigl\{i \in \mathbb{N}_0 : d\bigl(U_\gamma^{i+1}(x_t), U_\gamma^{i+1}(y)\bigr) \geqslant r\bigr\} \quad \text{and} \quad m \coloneqq \min\bigl\{i \in \mathbb{N}_0 : d\bigl(U_\gamma^{i}(x_t), U_\gamma^{i}(y)\bigr) \geqslant \rho\bigr\}, \tag{6.44}$$

noting that such N and m exist by Claim 2 and the fact that $\rho < r$ (cf. (6.42)). So Claim 2 gives

$$\bigl|U_\gamma^{i+1}(x_t) - U_\gamma^{i+1}(y)\bigr| = \gamma \bigl|U_\gamma^{i}(x_t) - U_\gamma^{i}(y)\bigr| = \gamma d\bigl(U_\gamma^i(x_t), \mathcal{O}_\gamma\bigr) \tag{6.45}$$

for all $0 \leqslant i \leqslant N$. Now $\gamma < \beta$, so combining (6.42) and (6.45), we conclude that

$$d\bigl(U_\gamma^N(x_t), \mathcal{O}_\gamma\bigr) = \gamma^{-1} \bigl|U_\gamma^{N+1}(x_t) - U_\gamma^{N+1}(y)\bigr| \geqslant \gamma^{-1} d\bigl(U_\gamma^{N+1}(x_t), \mathcal{O}_\gamma\bigr) \geqslant r/\gamma > r/\beta. \tag{6.46}$$

Now $d(x_t, \mathcal{O}_\gamma) < \rho$, so (6.46) and (6.42) imply that

$$1 \leqslant m \leqslant N. \tag{6.47}$$

In this case, we define $n_t \coloneqq N + 1$ and $x_{t+1} \coloneqq T_\gamma^{n_t}(x_t)$.

It is convenient to divide the following discussion into two subcases.

Subcase (i). Assume $B(\mathcal{O}_\gamma, \gamma\rho) \cap \mathcal{O}^*_\beta(1) = \emptyset$. In this subcase, by (6.47), (6.45), and the definition of m (cf. (6.44)), we have

$$|x_t - y| < \bigl|U_\gamma^m(x_t) - U_\gamma^m(y)\bigr| = \gamma \bigl|U_\gamma^{m-1}(x_t) - U_\gamma^{m-1}(y)\bigr| < \gamma\rho \leqslant d\bigl(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\bigr).$$

Then by (6.21), Theorem 4.9 (iii), and (6.14),

$$\begin{aligned}\left|S_m^{U_\gamma}\psi_\gamma(x_t)-S_m^{U_\gamma}\psi_\gamma(y)\right| &\leqslant \left|S_m^{U_\gamma}\overline{\phi}(x_t)-S_m^{U_\gamma}\overline{\phi}(y)\right|+|u(x_t)-u(y)|+\left|u\big(U_\gamma^m(x)\big)-u\big(U_\gamma^m(y)\big)\right| \\ &\leqslant |\phi|_\alpha\left(\sum_{i=0}^{m-1}\left|U_\gamma^i(x_t)-U_\gamma^i(y)\right|^\alpha+2K_\beta\rho^\alpha\gamma^\alpha\right) \\ &\leqslant |\phi|_\alpha\left(\left|U_\gamma^{m-1}(x_t)-U_\gamma^{m-1}(y)\right|^\alpha\sum_{i=0}^{m-1}\gamma^{-i\alpha}+2K_\beta\rho^\alpha\beta^\alpha\right) \\ &< |\phi|_\alpha\rho^\alpha\big(1+(\gamma^\alpha-1)^{-1}+2K_\beta\beta^\alpha\big) \\ &< |\phi|_\alpha\rho^\alpha L_1.\end{aligned} \tag{6.48}$$

Recalling that p denotes the common cardinality of $\mathcal{O}_\beta$ and $\mathcal{O}_\gamma$ (cf. Corollary 6.2 (i) and (6.11)), let us write $m=qp+l$, where q, l are integers with $q\geqslant 0$ and $l\in[0,p-1]$. Thus, since $\psi_\gamma\leqslant\tau$ by Claim 3, and $\eta=\int\phi'_\gamma\,\mathrm{d}\mu_{\mathcal{O}_\gamma}=\int\overline{\phi}\,\mathrm{d}\mu_{\mathcal{O}_\gamma}=\int\psi_\gamma\,\mathrm{d}\mu_{\mathcal{O}_\gamma}=p^{-1}S_p^{U_\gamma}\psi_\gamma(y)$, we have

$$S_m^{U_\gamma}\psi_\gamma(y)=qS_p^{U_\gamma}\psi_\gamma(y)+S_l^{U_\gamma}\psi_\gamma(y)\leqslant qp\eta+l\tau=m\eta+l(\tau-\eta)\leqslant m\eta+p(\tau-\eta), \tag{6.49}$$

where the final inequality uses that $\tau-\eta\geqslant 0$ (cf. (6.36)).

The two preceding inequalities (6.48) and (6.49), together with the definition of ρ (cf. (6.38)), and the fact that $C_1\geqslant 1$ and $\beta-\gamma<C_2<1$, give

$$\begin{aligned}S_m^{U_\gamma}\psi_\gamma(x_t) &\leqslant m\eta+p(\tau-\eta)+|\phi|_\alpha\rho^\alpha L_1=m\eta+(\tau-\eta)\big(p+L_1C_1^{-1}(\beta-\gamma)^{-\alpha/2}\big) \\ &\leqslant m\eta+(\tau-\eta)(\beta-\gamma)^{-\alpha/2}(p+L_1)\leqslant m\eta+(p+L_1)L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2},\end{aligned} \tag{6.50}$$

where in the final inequality we used that $\tau-\eta\leqslant L_2|\phi|_\alpha(\beta-\gamma)^\alpha$ (cf. (6.35)).

Now $n_t=N+1$, so $S_{n_t}^{U_\gamma}\psi'_\gamma(x_t)=S_m^{U_\gamma}\psi'_\gamma(x_t)+S_{N-m}^{U_\gamma}\psi'_\gamma\big(U_\gamma^m(x_t)\big)+\psi'_\gamma\big(U_\gamma^N(x_t)\big)$, and since $\psi'_\gamma\leqslant\psi_\gamma$ (cf. (6.33)) and $\psi'_\gamma\big(U_\gamma^i(x_t)\big)\leqslant\eta$ for $m\leqslant i<N$ by (6.39), then

$$S_{n_t}^{U_\gamma}\psi'_\gamma(x_t)\leqslant S_m^{U_\gamma}\psi_\gamma(x_t)+(N-m)\eta+\psi'_\gamma\big(U_\gamma^N(x_t)\big). \tag{6.51}$$

Now $\psi'_\gamma=\psi_\gamma-C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}d(\cdot,\mathcal{O}_\gamma)^\alpha$ (cf. (6.33)), so the fact that $d\big(U_\gamma^N(x_t),\mathcal{O}_\gamma\big)>r/\beta$ by (6.46), and that $\psi_\gamma\leqslant\tau$ by Claim 3, together give

$$\psi'_\gamma\big(U_\gamma^N(x_t)\big)\leqslant\tau-C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}(r/\beta)^\alpha,$$

and combining with (6.51) gives

$$S_{n_t}^{U_\gamma}\psi'_\gamma(x_t)\leqslant S_m^{U_\gamma}\psi_\gamma(x_t)+(N-m)\eta+\tau-C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}r^\alpha\beta^{-\alpha}. \tag{6.52}$$

Combining (6.50) and (6.52) gives

$$S_{n_t}^{U_\gamma}\psi'_\gamma(x_t)\leqslant m\eta+(p+L_1)L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2}+(N-m)\eta+\tau-C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2}r^\alpha\beta^{-\alpha},$$

and therefore

$$\begin{aligned}S_{n_t}^{U_\gamma}\psi'_\gamma(x_t)-n_t\eta &\leqslant (p+L_1)L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2}+(\tau-\eta)-C_1|\phi|_\alpha r^\alpha\beta^{-\alpha}(\beta-\gamma)^{\alpha/2} \\ &\leqslant (p+1+L_1)L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2}-C_1|\phi|_\alpha r^\alpha\beta^{-\alpha}(\beta-\gamma)^{\alpha/2}\leqslant 0,\end{aligned} \tag{6.53}$$

where the final two inequalities follow from the fact that $\tau-\eta\leqslant L_2|\phi|_\alpha(\beta-\gamma)^\alpha\leqslant L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2}$ (by (6.35), and since $\beta-\gamma<C_2<1$), and $C_1\geqslant r^{-\alpha}\beta^\alpha(p+1+L_1)L_2$ (cf. (6.16)).

Subcase (ii). Assume that $B(\mathcal{O}_\gamma,\gamma\rho)\cap\mathcal{O}_\beta^*(1)\neq\emptyset$. Now $\gamma<\beta$, so (6.42) and Claim 1 give

$$\gamma\rho<\beta\rho\leqslant r\leqslant r_\gamma, \tag{6.54}$$

which implies that

$$\mathcal{O}_\beta\cap\mathcal{O}_\beta^*(1)\neq\emptyset, \tag{6.55}$$

since if (6.55) were false then the definition of r_γ (cf. (6.20)) would give $r_\gamma \leqslant d(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1))$, and combining with (6.54) would yield $d(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)) > \gamma\rho$, contradicting the assumption of this subcase.

Now (6.55) means that

$$\mathcal{O}_\beta \subseteq \mathcal{O}^*_\beta(1),$$

in other words, $\mathcal{O}_\beta$ is precisely the U_β-periodic orbit corresponding to the eventually periodic critical orbit $\mathcal{O}^*_\beta(1)$. In particular, $\mathcal{O}_\beta \neq \{0\}$, since 0 is never in the critical orbit $\mathcal{O}^*_\beta(1)$ (as the only U_β-preimage of 0 is 0 itself). Recalling that $s > 0$ is the constant obtained from Corollary 6.2 (iii) when $\mathcal{O}_\beta \neq \{0\}$, then for each $x \in \mathcal{O}_\beta$, we obtain from Corollary 6.2 that

$$s(\beta - \gamma) \leqslant |(h_\gamma \circ \pi_\beta)(x) - x| \leqslant M(\beta - \gamma) < C_2 M \leqslant \frac{1}{2}\Delta\big(\mathcal{O}^*_\beta(1)\big), \tag{6.56}$$

using that $\beta - \gamma < C_2$ and $C_2 \leqslant \frac{1}{2M}\Delta\big(\mathcal{O}^*_\beta(1)\big)$ (by (6.13)). A consequence of the bound $|(\pi_\gamma \circ h_\beta)(x) - x| \leqslant \frac{1}{2}\Delta\big(\mathcal{O}^*_\beta(1)\big)$ from (6.56) is that if $x' \in \mathcal{O}_\gamma$, then the point $x \in \mathcal{O}_\beta$ satisfying $x' = (h_\gamma \circ \pi_\beta)(x) \in \mathcal{O}_\gamma$ must be the closest point in $\mathcal{O}^*_\beta(1)$ to x', so that $|x - x'| = d\big(x', \mathcal{O}^*_\beta(1)\big)$. This, together with (6.56), implies that

$$d\big(\mathcal{O}_\gamma, \mathcal{O}^*_\beta(1)\big) \geqslant s(\beta - \gamma), \tag{6.57}$$

which is a key estimate for this subcase, facilitating the following analysis.[29]

Now define

$$\delta := \beta^{-1} s(\beta - \gamma), \tag{6.58}$$

so that (6.57) and (6.58) give

$$B(\mathcal{O}_\gamma, \beta\delta) \cap \mathcal{O}^*_\beta(1) = \emptyset, \tag{6.59}$$

and since the assumption of this subcase is that $B(\mathcal{O}_\gamma, \gamma\rho) \cap \mathcal{O}^*_\beta(1) \neq \emptyset$, comparison with (6.59) yields

$$\delta < (\gamma/\beta)\rho < \rho. \tag{6.60}$$

Recalling from (6.43) that $y \in \mathcal{O}_\gamma$ is defined to satisfy $|x_t - y| = d(x_t, \mathcal{O}_\gamma)$, we now define

$$k := \min\big\{i \in \mathbb{N}_0 : d\big(U^i_\gamma(x_t), U^i_\gamma(y)\big) \geqslant \delta\big\} \in \mathbb{N}_0. \tag{6.61}$$

In particular, (6.60) and (6.61) imply that $0 \leqslant k \leqslant m$, since m was defined (cf. (6.44)) to be the smallest integer such that $d\big(U^m_\gamma(x_t), \mathcal{O}_\gamma\big) \geqslant \rho$. Note that $\rho/\delta > 1$ by (6.60), $\beta - C_2 \geqslant 1$ since $C_2 \leqslant c < \beta - 1$, and $0 \leqslant k \leqslant m \leqslant N$ by (6.47). By (6.45) and the definitions of m (cf. (6.44)) and k (cf. (6.61)), we have

$$m \leqslant k + \lceil \log_\gamma(\rho/\delta) \rceil \leqslant k + 1 + \log_{\beta - C_2}(\rho/\delta). \tag{6.62}$$

Combining (6.62) with the definitions $\rho = (\tau - \eta)^{1/\alpha}(C_1|\phi|_\alpha)^{-1/\alpha}(\beta - \gamma)^{-1/2}$ (cf. (6.38)), and $\delta = \beta^{-1}s(\beta - \gamma)$ (cf. (6.58)), and the bound $\tau - \eta \leqslant L_2|\phi|_\alpha(\beta - \gamma)^\alpha$ (cf. (6.35)), we see that

$$m \leqslant k + 1 + \frac{1}{\ln(\beta - C_2)}\Big(\ln\beta + \frac{1}{\alpha}(\ln L_2 - \ln C_1) - \ln s - \frac{1}{2}\ln(\beta - \gamma)\Big). \tag{6.63}$$

Now $C_1 \geqslant 1$ (cf. (6.16)) and $L_3 = 1 + \frac{1}{\ln(\beta - C_2)}\big(\ln\beta + \frac{1}{\alpha}\ln L_2 - \ln s\big)$ (cf. (6.17)), so (6.63) implies that

$$m \leqslant k + L_3 - \frac{\ln(\beta - \gamma)}{2\ln(\beta - C_2)}. \tag{6.64}$$

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[29] Some remarks on the differences between Subcases (i) and (ii) are in order; in particular, although the proof strategy for Subcase (i) resembles arguments used in Section 2, Subcase (ii) has no such analogue. While in Subcase (i), ψ_γ is α-Hölder on $B(\mathcal{O}_\gamma, \rho)$, so that the length-$m$ orbit sums in (6.48) can be bounded in terms of $\big|U^i_\gamma(x_t) - U^i_\gamma(y)\big|$, the same is not true in Subcase (ii), and since m could be arbitrarily large (as the distance $|x_t - y|$ could be arbitrarily small), the crude bound $\psi_\gamma\big(U^i_\gamma(x_t)\big) \leqslant \tau$ would then be insufficient to obtain an effective bound analogous to (6.50). However, (6.57) allows us to use estimates analogous to (6.48) for the first k terms in the length-m orbit sum, while the (proportionally small number of) remaining terms can be crudely bounded using $\psi_\gamma\big(U^i_\gamma(x_t)\big) \leqslant \tau$, and the corresponding term $(m - k)(\tau - \eta)$ in (6.68) can be estimated by a multiple of $(\beta - \gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha$.

In the case where $k \geqslant 1$, the definition of k in (6.61), together with $k \leqslant m \leqslant N$, (6.45), and (6.59), gives

$$|x_t - y| < \delta \leqslant \big|U_\gamma^k(x_t) - U_\gamma^k(y)\big| = \gamma\big|U_\gamma^{k-1}(x_t) - U_\gamma^{k-1}(y)\big| < \gamma\delta < \beta\delta \leqslant d\big(\mathcal{O}_\gamma, \mathcal{O}_\beta^*(1)\big).$$

We now wish to estimate orbit sums for the function ψ_γ. By exactly the same reasoning as was used to derive (6.48) in Subcase (i) (except that here δ replaces the ρ in (6.48)), we obtain

$$\big|S_k^{U_\gamma}\psi_\gamma(x_t) - S_k^{U_\gamma}\psi_\gamma(y)\big| \leqslant |\phi|_\alpha \delta^\alpha L_1. \tag{6.65}$$

Similarly, the arguments used to derive (6.49) in Subcase (i) can be used here (with k replacing the m used in (6.49)) to obtain

$$S_k^{U_\gamma}\psi_\gamma(y) \leqslant k\eta + p(\tau - \eta). \tag{6.66}$$

Obviously, (6.65) and (6.66) also hold for $k = 0$.

Now $\psi_\gamma \leqslant \tau$ by Claim 3, so

$$S_{m-k}^{U_\gamma}\psi_\gamma\big(U_\gamma^k(x_t)\big) \leqslant (m-k)\tau. \tag{6.67}$$

Combining (6.66), (6.65), and (6.67) gives

$$\begin{aligned} S_m^{U_\gamma}\psi_\gamma(x_t) &\leqslant S_k^{U_\gamma}\psi_\gamma(y) + \big|S_k^{U_\gamma}\psi_\gamma(x_t) - S_k^{U_\gamma}\psi_\gamma(y)\big| + S_{m-k}^{U_\gamma}\psi_\gamma\big(U_\gamma^k(x_t)\big) \\ &\leqslant m\eta + |\phi|_\alpha\delta^\alpha L_1 + p(\tau-\eta) + (m-k)(\tau-\eta), \end{aligned} \tag{6.68}$$

and using (6.58) and (6.64) to estimate the right-hand side of (6.68) yields

$$S_m^{U_\gamma}\psi_\gamma(x_t) \leqslant m\eta + |\phi|_\alpha \frac{s^\alpha}{\beta^\alpha}(\beta-\gamma)^\alpha L_1 + \left(p + L_3 - \frac{\ln(\beta-\gamma)}{2\ln(\beta - C_2)}\right)(\tau - \eta). \tag{6.69}$$

But $\tau - \eta \leqslant L_2|\phi|_\alpha(\beta-\gamma)^\alpha$ by (6.35), so (6.69) gives

$$S_m^{U_\gamma}\psi_\gamma(x_t) \leqslant m\eta + |\phi|_\alpha(\beta-\gamma)^\alpha\left(\frac{s^\alpha}{\beta^\alpha}L_1 + pL_2 + L_3L_2 - \frac{L_2\ln(\beta-\gamma)}{2\ln(\beta-C_2)}\right). \tag{6.70}$$

Now $(\beta-\gamma)^\alpha \leqslant (\beta-\gamma)^{\alpha/2}$, and we can use the fact[30] that $x^{\alpha/2}\ln x \geqslant -\frac{2}{e\alpha}$ for all $x > 0$ to derive $-(\beta-\gamma)^\alpha\ln(\beta-\gamma) \leqslant \frac{2(\beta-\gamma)^{\alpha/2}}{e\alpha}$, so (6.70) implies

$$\begin{aligned} S_m^{U_\gamma}\psi_\gamma(x_t) &\leqslant m\eta + |\phi|_\alpha(\beta-\gamma)^{\alpha/2}\left(\frac{s^\alpha}{\beta^\alpha}L_1 + pL_2 + L_3L_2 + \frac{L_2}{e\alpha\ln(\beta-C_2)}\right) \\ &= m\eta + |\phi|_\alpha(\beta-\gamma)^{\alpha/2}L_4, \end{aligned} \tag{6.71}$$

where we use that $L_4 = \frac{s^\alpha}{\beta^\alpha}L_1 + pL_2 + L_3L_2 + \frac{L_2}{e\alpha\ln(\beta-C_2)}$ (cf. (6.18)).

Now we wish to argue that $S_{n_t}^{U_\gamma}\psi'_\gamma(x_t) \leqslant n_t\eta$, and will do so in a way that is analogous to the derivation of (6.53) in Subcase (i). Specifically, we use that $\psi'_\gamma \leqslant \psi_\gamma$ by (6.33), $\psi'_\gamma\big(U_\beta^i(x_t)\big) \leqslant \eta$ for $m \leqslant i < N$ by (6.39) (cf. (6.44)), and $d\big(U_\gamma^N(x_t), \mathcal{O}_\gamma\big) > r/\beta$ (cf. (6.46)), together with (6.71) above, the definition (6.33) of ψ'_γ, Claim 3, the bound $\tau - \eta \leqslant L_2|\phi|_\alpha(\beta-\gamma)^\alpha \leqslant L_2|\phi|_\alpha(\beta-\gamma)^{\alpha/2}$ (cf. (6.35)), and the fact that $C_1 \geqslant r^{-\alpha}\beta^\alpha(L_4 + L_2)$ (which is valid since $\mathcal{O}_\beta \neq \{0\}$, cf. (6.19)), to see that

$$\begin{aligned} S_{n_t}^{U_\gamma}\psi'_\gamma(x_t) - n_t\eta &\leqslant S_m^{U_\gamma}\psi_\gamma(x_t) - m\eta + S_{N-m}^{U_\gamma}\psi'_\gamma\big(U_\gamma^m(x_t)\big) - (N-m)\eta + \psi'_\gamma\big(U_\gamma^N(x_t)\big) - \eta \\ &\leqslant L_4|\phi|_\alpha(\beta-\gamma)^{\alpha/2} + (\tau-\eta) - C_1|\phi|_\alpha r^\alpha\beta^{-\alpha}(\beta-\gamma)^{\alpha/2} \\ &\leqslant (L_4 + L_2)|\phi|_\alpha(\beta-\gamma)^{\alpha/2} - C_1|\phi|_\alpha r^\alpha\beta^{-\alpha}(\beta-\gamma)^{\alpha/2} \\ &\leqslant 0. \end{aligned} \tag{6.72}$$

This completes Subcase (ii), and therefore Case C is complete. Having concluded each of Cases A, B, and C, the recursive step is now complete.

[30]It is readily shown that $-\frac{2}{e\alpha}$ is the minimum of the function $x \mapsto x^{\alpha/2}\ln x$.

The four inequalities (6.40), (6.41), (6.53), and (6.72) mean that $S_{n_k}^{U_\gamma}\psi'_\gamma(x_k) \leqslant n_k\eta$ for all $k \in \mathbb{N}$. Therefore, defining $N_k := n_1 + \cdots + n_k$ for each $k \in \mathbb{N}$, we have

$$\liminf_{n\to+\infty} \frac{1}{n} S_n^{U_\gamma}\psi'_\gamma(x) \leqslant \liminf_{k\to+\infty} \frac{1}{N_k}\sum_{i=1}^{k} S_{n_i}^{U_\gamma}\psi'_\gamma(x_i) \leqslant \liminf_{k\to+\infty} \frac{1}{N_k}\sum_{i=1}^{k} n_i\eta = \eta.$$

But this is precisely the required inequality (6.37), and therefore, as noted prior to the statement of (6.37), we deduce that $\mu_{\mathcal{O}_\gamma}$ is (U_γ, ϕ'_γ)-maximizing. In other words, the periodic measure $\mu_{\mathcal{O}_\gamma}$ is U_γ-maximizing for the function

$$\phi'_\gamma = \overline{\phi} - C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha. \tag{6.73}$$

Now the function

$$-C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha \tag{6.74}$$

attains its maximum value precisely on the set $\mathcal{O}_\gamma$, so its *unique* U_γ-maximizing measure is $\mu_{\mathcal{O}_\gamma}$. It follows that the sum of the functions (6.73) and (6.74), namely the function

$$\overline{\phi} - 2C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha,$$

has $\mu_{\mathcal{O}_\gamma}$ as its unique U_γ-maximizing measure. Consequently, the function defined (cf. (6.8)) in the statement of the theorem, namely

$$\phi - 2C_1|\phi|_\alpha(\beta-\gamma)^{\alpha/2} d(\cdot, \mathcal{O}_\gamma)^\alpha,$$

also has $\mu_{\mathcal{O}_\gamma}$ as its unique U_γ-maximizing measure, and this completes the proof of part (ii) of the theorem. □

6.3. **Proof of Theorem C (Joint TPO for beta-transformations).** In order to prove Theorem C′ (and hence Theorem C), a final ingredient is that the set of non-simple beta-numbers is dense in $(1, +\infty)$. This complement to Parry's result on the density of simple beta-numbers [Pa60, Theorem 5] does not seem to be available in the literature, so we prove it here:

Lemma 6.5. *The set of non-simple beta-numbers is a dense subset of* $(1, +\infty)$.

Proof. Since the set of simple beta-numbers is countable, it suffices to show that if $1 < \beta_1 < \beta_2$, where neither β_1 nor β_2 is a simple beta-number, then there is a non-simple beta-number $\gamma \in (\beta_1, \beta_2)$.

Writing

$$\pi_{\beta_1}(1) = a_1a_2\dots \quad \text{and} \quad \pi_{\beta_2}(1) = b_1b_2\dots,$$

the fact that $\beta_1 < \beta_2$ means, by Proposition 3.9 (xiii), that $\pi_{\beta_1}(1) \prec \pi_{\beta_2}(1)$. Hence, there exists $m \in \mathbb{N}$ with $a_1 \dots a_m \prec b_1 \dots b_m$, and $b_m \neq 0$. Since β_2 is not a simple beta-number, it follows that $b_{m+1}b_{m+2}\dots \neq (0)^\infty$, so there exists $n \in \mathbb{N}$ such that $b_1 \dots b_m(0)^n \prec b_1 \dots b_{m+n}$, and this n can be chosen such that $n \geqslant m$. Defining

$$\underline{c} := b_1 \dots b_m (0)^n \big(a_1(0)^{m-1}\big)^\infty,$$

we see that $\pi_{\beta_1}(1) \prec \underline{c} \prec \pi_{\beta_2}(1)$, and claim that there exists a non-simple beta-number $\gamma \in (\beta_1, \beta_2)$ such that $\pi_\gamma(1) = \underline{c}$.

Recall that by [Pa60, Corollary 1], to prove that there exists $\gamma > 1$ with $\pi_\gamma(1) = \underline{c}$, it suffices to show that $\sigma^t(\underline{c}) \prec \underline{c}$ for all $t \in \mathbb{N}$. For this, note first that $a_1(0)^{m-1} \preceq a_1 \dots a_m \prec b_1 \dots b_m$, and $b_1 > 0$, so if $t \geqslant m$ then $\sigma^t(\underline{c}) \prec \underline{c}$. On the other hand, if $t < m$ then

$$b_{t+1} \dots b_m (0)^t \preceq b_{t+1} \dots b_{t+m} \preceq b_1 \dots b_m, \tag{6.75}$$

the second inequality in (6.75) following from the fact that $\sigma^t(\pi_{\beta_2}(1)) \prec \pi_{\beta_2}(1)$ (cf. [Pa60, Theorem 3]). Now $b_m \neq 0$, so (6.75) implies the strict inequality

$$b_{t+1} \dots b_m (0)^t \prec b_1 \dots b_m. \tag{6.76}$$

The fact that $n \geqslant m > t$ implies that $b_{t+1} \dots b_m (0)^t$ are the first m terms of $\sigma^t(\underline{c})$, and since moreover $b_1 \dots b_m$ are the first m terms of $\underline{c}$, (6.76) implies that $\sigma^t(\underline{c}) \prec \underline{c}$. Having shown that $\sigma^t(\underline{c}) \prec \underline{c}$ for all $t \in \mathbb{N}$, it follows that there exists $\gamma > 1$ with $\pi_\gamma(1) = \underline{c}$.

Now $\underline{c}$ is pre-periodic under σ, so γ is a beta-number, and $\underline{c}$ has nonzero tails since $a_1 > 0$, thus γ is a non-simple beta-number. Finally, the fact that $\pi_{\beta_1}(1) \prec \underline{c} \prec \pi_{\beta_2}(1)$ implies, by Proposition 3.9 (xiii), that $\gamma \in (\beta_1, \beta_2)$. □

We are now able to prove the joint typical periodic optimization theorem using the results above: the following Theorem C′ represents a slightly stronger version of Theorem C:

Theorem C′ (Joint TPO for beta-transformations and upper beta-transformations). *Given* $\alpha \in (0,1]$, *the sets* $\mathfrak{L}_U^\alpha := \{(\beta,\phi) \in (1,+\infty) \times C^{0,\alpha}(I) : \phi \in \mathrm{Lock}^\alpha(U_\beta)\}$ *and* $\mathfrak{L}_T^\alpha := \{(\beta,\phi) \in (1,+\infty) \times C^{0,\alpha}(I) : \phi \in \mathrm{Lock}^\alpha(T_\beta)\}$ *both contain an open and dense subset of* $(1,+\infty) \times C^{0,\alpha}(I)$.

Proof. We first prove the result for $\mathfrak{L}_U^\alpha$. Suppose $\beta \in (1,+\infty)$, $\phi \in C^{0,\alpha}(I)$, and let $\varepsilon > 0$ be arbitrary. Note that the set of non-simple beta-numbers is dense in $(1,+\infty)$, by Lemma 6.5, so there exists a non-simple beta-number $\theta > 1$ with $|\theta - \beta| < \varepsilon$. By Theorem G′ (Individual TPO theorem for beta-numbers), applied to θ, there exists $\Phi \in \mathrm{Lock}^\alpha(U_\theta)$ with $\|\phi - \Phi\|_\alpha < \varepsilon$. If $\mathcal{O}_\theta$ denotes the unique maximizing periodic orbit for Φ, then there exists $\delta > 0$ such that $\mathcal{O}_\theta$ is the unique maximizing periodic orbit for each function in $B(\Phi,\delta)$ (with respect to $\|\cdot\|_\alpha$), and we may assume that $\delta < (1/2)|\Phi|_\alpha$. Let $\psi \in B(\Phi,\delta)$. Thus, we have $|\Phi - \psi|_\alpha \leqslant \|\Phi - \psi\|_\alpha \leqslant \delta \leqslant (1/2)|\Phi|_\alpha$. As $|\cdot|_\alpha$ is sub-additive, we get

$$\frac{1}{2}|\Phi|_\alpha \leqslant |\psi|_\alpha \leqslant \frac{3}{2}|\Phi|_\alpha. \tag{6.77}$$

Now let C_1, C_2 denote the constants obtained by applying Theorem 6.4 (ii) to θ and $\mathcal{O}_\theta$, and set

$$E := \min\{C_2,\, (\delta/(36C_1|\Phi|_\alpha))^{2/\alpha}\}. \tag{6.78}$$

For each $\gamma \in (\theta - E, \theta)$ and each $\psi \in B(\Phi, \delta/2)$, we set $\mathcal{O}_\gamma := (h_\gamma \circ \pi_\theta)(\mathcal{O}_\theta)$ and

$$\psi' := \psi + 6C_1|\psi|_\alpha(\theta-\gamma)^{\alpha/2} d(\cdot,\mathcal{O}_\gamma)^\alpha. \tag{6.79}$$

Note that

$$\|d(\cdot,\mathcal{O}_\gamma)^\alpha\|_\alpha = |d(\cdot,\mathcal{O}_\gamma)^\alpha|_\alpha + \|d(\cdot,\mathcal{O}_\gamma)^\alpha\|_\infty \leqslant 2.$$

Then from $|\theta - \gamma| < E$, (6.78), and (6.79), we conclude that

$$\|\psi' - \psi\|_\alpha \leqslant 12C_1|\psi|_\alpha E^{\alpha/2} \leqslant 18C_1|\Phi|_\alpha E^{\alpha/2} \leqslant \delta/2,$$

so that $\psi' \in B(\Phi,\delta) \subseteq \mathrm{Lock}^\alpha(U_\theta)$. Thus, applying Theorem 6.4 (ii) to θ, $\mathcal{O}_\theta$, and ψ', we obtain that $\mu_{\mathcal{O}_\gamma}$ uniquely maximizes $\psi' - 2C_1|\psi'|_\alpha(\theta-\gamma)^{\alpha/2}d(\cdot,\mathcal{O}_\gamma)^\alpha$. As ψ, $\psi' \in B(\Phi,\delta)$, applying (6.77) gives $6C_1|\psi|_\alpha \geqslant 3C_1|\Phi|_\alpha \geqslant 2C_1|\psi'|_\alpha$, so the measure $\mu_{\mathcal{O}_\gamma}$ also uniquely maximizes the function $\psi = \psi' - 6C_1|\psi|_\alpha(\theta-\gamma)^{\alpha/2}d(\cdot,\mathcal{O}_\gamma)^\alpha$. Therefore, $(\theta - E, \theta) \times B(\Phi,\delta/2)$ is contained in the interior of the set $\mathfrak{L}_U^\alpha$. Since $\varepsilon > 0$ was arbitrary, this completes the proof for the set $\mathfrak{L}_U^\alpha$.

It remains to prove the assertion about $\mathfrak{L}_T^\alpha$. Now non-simple beta-numbers are non-emergent by Corollary 5.4, and $\mathrm{Lock}^\alpha(T_\beta)$ is an open and dense subset of $C^{0,\alpha}(I)$ when $\beta > 1$ is a non-simple beta-number by Theorem H′, so an argument analogous to the one above, using part (i) (rather than part (ii)) of Theorem 6.4, can be used to show that $\mathfrak{L}_T^\alpha$ contains an open dense subset of $(1,+\infty) \times C^{0,\alpha}(I)$, as required. □

Having proved the Joint TPO result Theorem C′, we can now deduce Theorem F. We establish the following slightly stronger version of Theorem F (which in particular implies Theorem F):

Theorem F′ (Individual TPO for generic potentials). *Fix* $\alpha \in (0,1]$. *There is a residual subset* $R \subseteq C^{0,\alpha}(I)$ *such that for all* $\phi \in R$, *there is an open and dense set of parameters* $B_\phi \subseteq (1,+\infty)$ *such that* $\phi \in \mathrm{Lock}^\alpha(T_\beta)$ *for all* $\beta \in B_\phi$.

Proof. The set $\mathfrak{L}^\alpha_T = \{(\beta,\phi) \in (1,+\infty) \times C^{0,\alpha}(I) : \phi \in \mathrm{Lock}^\alpha(T_\beta)\}$ contains an open dense subset of $(1,+\infty) \times C^{0,\alpha}(I)$, by Theorem C′, and the space $(1,+\infty)$ is second countable. A standard result from topology (see e.g. [Ke95, Lemma 8.42]) then implies that there is a residual subset $R \subseteq C^{0,\alpha}(I)$ such that if $\phi \in R$ then $\{\beta \in (1,+\infty) : \phi \in \mathrm{Lock}^\alpha(T_\beta)\}$ contains an open and dense subset of $(1,+\infty)$, as required. □

Appendix A. Beta-transformations and maximizing measures: proofs

The purpose of this appendix is to prove the results stated in Section 3.

Proof of Lemma 3.4. Define functions $f, g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f(u) \coloneqq \beta u, \quad g_1(u) \coloneqq u - \lfloor u \rfloor, \quad g_2(u) \coloneqq u - \lfloor u \rfloor'.$$

Then f is continuous and strictly increasing, and for each $u \in \mathbb{R}$, we have

$$\lim_{x\nearrow u} g_1(x) = g_2(u)^-, \quad \lim_{x\searrow u} g_1(x) = g_1(u)^+, \quad \lim_{x\nearrow u} g_2(x) = g_2(u)^-. \tag{A.1}$$

(i) By (3.1), we have $T_\beta = g_1 \circ f$ and $T^n_\beta = g_1 \circ f \circ T^{n-1}_\beta$. By (A.1), $\lim_{x\searrow a} T_\beta(x) = T_\beta(a)^+$. Assume that $\lim_{x\searrow a} T^n_\beta(x) = T^n_\beta(a)^+$ when $n = k$. When $n = k+1$,

$$\lim_{x\searrow a} T^{k+1}_\beta(x) = \lim_{x\searrow a} T_\beta\big(T^k_\beta(x)\big) = \lim_{y\searrow T^k_\beta(x)} T_\beta(y) = T^{k+1}_\beta(a)^+.$$

By induction, for all $n \in \mathbb{N}$, $\lim_{x\searrow a} T^n_\beta(x) = T^n_\beta(a)^+$. We can prove $\lim_{x\nearrow b} U^n_\beta(x) = U^n_\beta(b)^-$ similarly.

(ii) By (3.3) and (3.4), we have $\varepsilon_n(\cdot,\beta) = \lfloor\cdot\rfloor \circ f \circ T^{n-1}_\beta$ and $\varepsilon^*_n(\cdot,\beta) = \lfloor\cdot\rfloor' \circ f \circ U^{n-1}_\beta$. Hence (ii) follows from (i) and the fact that $\lfloor\cdot\rfloor$ is right-continuous and $\lfloor\cdot\rfloor'$ is left-continuous.

(iii) follows immediately from (3.1), (3.3), (3.4), and $U_\beta(0) = 0$.

(iv) Since $T_\beta = g_1 \circ f$ and $U_\beta = g_2 \circ f$ (see (3.1) and (3.2)), by (A.1),

$$\lim_{x\nearrow b} T_\beta(x) = \lim_{x\nearrow b} (g_1 \circ f)(x) = \lim_{u\nearrow f(b)} g_1(u) = (g_2 \circ f)(b)^- = U_\beta(b)^-.$$

Assume that $\lim_{x\nearrow b} T^n_\beta(x) = U^n_\beta(b)^-$ holds for $n = k$. When $n = k+1$,

$$\lim_{x\nearrow b} T^{k+1}_\beta(x) = \lim_{x\nearrow b} T_\beta\big(T^k_\beta(x)\big) = \lim_{u\nearrow U^k_\beta(b)} T_\beta(u) = U_\beta\big(U^k_\beta(b)\big)^- = U^{k+1}_\beta(b)^-.$$

Hence the first part of (iv) follows by induction.

By (3.3) and the first part of (iv),

$$\lim_{x\nearrow b} \varepsilon_n(x,\beta) = \lim_{x\nearrow b} (\lfloor\cdot\rfloor \circ f)\big(T^{n-1}_\beta(x)\big) = \lim_{u\nearrow U^{n-1}_\beta(b)} (\lfloor\cdot\rfloor \circ f)(u) = \lim_{v\nearrow f(U^{n-1}_\beta(b))} \lfloor v \rfloor.$$

By the fact that $\lim_{x\nearrow u} \lfloor x \rfloor = \lfloor u \rfloor'$ for all $u \in \mathbb{R}$ and (3.4),

$$\lim_{v\nearrow f(U^{n-1}_\beta(b))} \lfloor v \rfloor = \big\lfloor \beta U^{n-1}_\beta(b)) \big\rfloor' = \varepsilon^*_n(b,\beta).$$

The second part of (iv) follows from the above two equalities. □

Proof of Lemma 3.5. Without loss of generality we can assume that $x \neq 0$ (see Lemma 3.4 (iii)). Define functions $f, g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ by

$$f(u) \coloneqq xu, \quad g_1(u) \coloneqq u - \lfloor u \rfloor, \quad g_2(u) \coloneqq u - \lfloor u \rfloor'.$$

Note that f is continuous and strictly increasing and

$$\lim_{x\nearrow u} g_1(x) = g_2(u)^-, \quad \lim_{x\searrow u} g_1(x) = g_1(u)^+, \quad \lim_{x\nearrow u} g_2(x) = g_2(u)^-. \tag{A.2}$$

(i) By (3.1), we have $T_\beta(x) = (g_1 \circ f)(\beta)$ and $T^n_\beta(x) = g_1\big(\beta T^{n-1}_\beta(x)\big)$. By (A.2), $\lim_{\gamma\searrow\beta} T_\gamma(x) = T_\beta(x)^+$. Since $\beta > 1$, if $\lim_{\gamma\searrow\beta} T^{k-1}_\gamma(x) = T^{k-1}_\beta(x)^+$ for some $k \in \mathbb{N}$, we have $\lim_{\gamma\searrow\beta} \gamma T^{k-1}_\gamma(x) =$

$\beta T_\beta^{k-1}(x)^+$. Then by (A.2), $\lim_{\gamma\searrow\beta} T_\gamma^k(x) = T_\beta^k(x)^+$. By induction, for each $\beta > 1$, $\lim_{\gamma\searrow\beta} T_\gamma^n(x) = T_\beta^n(x)^+$. Similarly, we have $\lim_{\gamma\nearrow\beta} U_\beta^n(x) = U_\beta^n(x)^-$.

(ii) By (i), and the fact that $\beta > 1$, we have $\lim_{\gamma\searrow\beta} \gamma T_\gamma^{n-1}(x) = \beta T_\beta^{n-1}(x)^+$ and $\lim_{\gamma\nearrow\beta} \gamma U_\gamma^{n-1}(x) = \beta U_\beta^{n-1}(x)^-$. Since $\lfloor\cdot\rfloor$ is right-continuous and $\lfloor\cdot\rfloor'$ is left-continuous, (ii) follows from (3.3) and (3.4).

(iii) Since $T_\beta(x) = (g_1 \circ f)(\beta)$ and $U_\beta(x) = (g_2 \circ f)(\beta)$, by (A.2),

$$\lim_{\gamma\nearrow\beta} T_\gamma(x) = \lim_{\gamma\nearrow\beta} g_1(\gamma x) = \lim_{u\nearrow f(\beta)} g_1(u) = g_2(f(\beta))^- = U_\beta(x)^-.$$

Assume that $\lim_{\gamma\nearrow\beta} T_\gamma^n(x) = U_\beta^n(x)^-$ holds for $n = k$. When $n = k+1$, by (A.2),

$$\lim_{\gamma\nearrow\beta} T_\gamma^{k+1}(x) = \lim_{\gamma\nearrow\beta} g_1\bigl(\gamma T_\gamma^k(x)\bigr) = \lim_{u\nearrow\beta U_\beta^k(x)} g_1(u) = g_2\bigl(\beta U_\beta^k(x)\bigr)^- = U_\beta^{k+1}(x)^-.$$

Hence the first part of (iii) follows from induction.

By (3.3), (3.4), the first part of (iii), and the fact that $\lim_{x\nearrow u}\lfloor x\rfloor = \lfloor u\rfloor'$ for all $u \in \mathbb{R}$,

$$\lim_{\gamma\nearrow\beta} \varepsilon_n(x,\gamma) = \lim_{\gamma\nearrow\beta} \bigl\lfloor \gamma T_\gamma^{n-1}(x)\bigr\rfloor = \lim_{u\nearrow\beta U_\beta^{n-1}(x)} \lfloor u\rfloor = \bigl\lfloor \beta U_\beta^{n-1}(x)\bigr\rfloor' = \varepsilon_n^*(x,\beta).$$

Therefore, the second part of (iii) follows. □

Proof of Lemma 3.8. The first part follows from Lemma 3.4 (iii), whereas the second part follows from Lemma 3.4 (iv). □

Proof of Proposition 3.9. Statements (i), (ii), (viii), and (ix) follow from [YT21, Lemma 1.2].

The statements about π_β in (iv) and (vi) follow from [IT74, Proposition 3.2], while statements about π_β^* in (iv) and (vi) follow from [YT21, Lemma 1.2].

Statements (x) and (xi) follow from [IT74, Proposition 3.2].

It remains to prove statements (iii), (v), (vii), and (xii)–(xiv).

(iii) follows from (3.3), (3.4), and Definition 3.6.

(v) From (iii) and (iv), we have $h_\beta \circ \sigma \circ \pi_\beta = h_\beta \circ \pi_\beta \circ T_\beta = T_\beta = T_\beta \circ h_\beta \circ \pi_\beta$. Thus, $h_\beta \circ \sigma = T_\beta \circ h_\beta$ on $\pi_\beta(I)$. Similarly, we have $h_\beta \circ \sigma = U_\beta \circ h_\beta$ on $\pi_\beta^*(I)$.

(vii) Consider arbitrary $x, y \in I$ with $0 \leqslant x < y \leqslant 1$. By Lemma 3.8, we have $\lim_{z\nearrow y} \pi_\beta(z) = \pi_\beta^*(y)$. Combining this with the fact that π_β is strictly increasing (see (vi)), we obtain $\pi_\beta(x) \prec \pi_\beta^*(y)$.

(xii) Fix $\beta > 1$. Consider a pair of sequences $A = a_1a_2\dots$ and $B = b_1b_2\dots$ in X_β. Assume that $d_\beta(A,B) = \beta^{-k}$ for some integer $k \in \mathbb{N}$. By (3.7), we have

$$|h_\beta(A) - h_\beta(B)| \leqslant \sum_{n=k}^{+\infty} \frac{|a_n - b_n|}{\beta^n} \leqslant \beta \sum_{n=k}^{+\infty} \frac{1}{\beta^n} = \frac{1}{\beta^{k-2}(\beta-1)} = \frac{\beta^2}{\beta-1} d_\beta(A,B).$$

(xiii) By (3.8) and Lemma 3.8, $i_0(\beta) = i_0^*(\beta) = (0)^\infty$ for all $\beta > 1$.

Fix arbitrary $x \in (0,1]$. It is easy to see that $i_x(\beta_1) \neq i_x(\beta_2)$ when $\beta_1 \neq \beta_2$. So it suffices to prove that i_x is non-decreasing. Assume that there exist $1 < \beta_1 < \beta_2$ such that $i_x(\beta_2) \prec i_x(\beta_1)$. Then there exists $n \in \mathbb{N}$ such that $\varepsilon_k(\beta_1,x) = \varepsilon_k(\beta_2,x)$ for all $k \in \{1, \dots, n-1\}$ and $\varepsilon_n(\beta_1,x) > \varepsilon_n(\beta_2,x)$. Then we have $\varepsilon_n(\beta_1,x) \geqslant \varepsilon_n(\beta_2,x) + 1 \geqslant 1$. By (iii) and (iv), for each $\beta > 1$, we have

$$x = h_\beta(\pi_\beta(x)) = \sum_{k=1}^{n-1} \frac{\varepsilon_k(x,\beta)}{\beta^k} + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \frac{T_\beta^n(x)}{\beta^n}.$$

So we obtain

$$x = \sum_{k=1}^{n-1} \frac{\varepsilon_k(x,\beta_2)}{\beta_2^k} + \frac{\varepsilon_n(x,\beta_2)}{\beta_2^n} + \frac{T_{\beta_2}^n(x)}{\beta_2^n} \leqslant \sum_{k=1}^{n-1} \frac{\varepsilon_k(x,\beta_2)}{\beta_2^k} + \frac{\varepsilon_n(x,\beta_2)+1}{\beta_2^n}$$
$$\leqslant \sum_{k=1}^{n} \frac{\varepsilon_k(x,\beta_1)}{\beta_2^k} < \sum_{k=1}^{n} \frac{\varepsilon_k(x,\beta_1)}{\beta_1^k} \leqslant x,$$

which leads to a contradiction. So i_x is strictly increasing. The proof that i_x^* is strictly increasing follows by using a similar argument.

(xiv) Fix $x \in I$. By Lemma 3.5 (ii), we have that $\varepsilon_n(x,\cdot)$ is right-continuous and $\varepsilon_n^*(x,\cdot)$ is left-continuous for each $n \in \mathbb{N}$. By Definition 3.6 and (3.8), (xiv) follows. $\square$

Proof of Proposition 3.12. (i) These properties follow immediately from the definitions of T_β, U_β, and D_β.

(ii) Fix a periodic orbit $\mathcal{O}_\beta$ of T_β that is not $\{0\}$. By (i) we have $1 \notin \mathcal{O}_\beta$ and $\mathcal{O}_\beta \cap D_\beta = \emptyset$. Then $U_\beta(x) = T_\beta(x)$ for all $x \in \mathcal{O}_\beta$. Hence $\mathcal{O}_\beta$ is also a periodic orbit of U_β.

Fix a periodic orbit $\mathcal{O}_\beta^*$ of U_β. If $1 \in \mathcal{O}_\beta^*$, by (i) we get that $\mathcal{O}_\beta^*$ is not a periodic orbit of T_β. If $1 \notin \mathcal{O}_\beta^*$, by (i) we have $\mathcal{O}_\beta^* \cap D_\beta = \emptyset$. Then we have $U_\beta(x) = T_\beta(x)$ for all $x \in \mathcal{O}_\beta^*$. Hence $\mathcal{O}_\beta^*$ is a periodic orbit for T_β.

(iii) Fix an arbitrary $\mu \in \mathcal{M}(I, T_\beta)$. By (i) we have $\mu(\{1\}) = \mu\bigl(T_\beta^{-1}(1)\bigr) = \mu(\emptyset) = 0$ and $\mu(D_\beta) = \mu\bigl(T_\beta^{-1}(0)\bigr) - \mu(\{0\}) = 0$. Then we have $\mu\bigl(U_\beta^{-1}(0)\bigr) = \mu(\{0\})$ and $\mu\bigl(U_\beta^{-1}(1)\bigr) = \mu(D_\beta) = 0 = \mu(\{1\})$. By definition we have $T_\beta^{-1}(Y) = U_\beta^{-1}(Y)$ for all Borel measurable subsets $Y \subseteq (0,1)$. Hence, we have $\mu \in \mathcal{M}(I, U_\beta)$.

Now fix an arbitrary $\nu \in \mathcal{M}(I, U_\beta)$. If $\nu(\{1\}) = 0$, then by (i) we have $\nu\bigl(T_\beta^{-1}(1)\bigr) = \nu(\emptyset) = 0 = \nu(\{1\})$, $\nu(D_\beta) = \nu\bigl(U_\beta^{-1}(1)\bigr) = \nu(\{1\}) = 0$, and $\nu\bigl(T_\beta^{-1}(0)\bigr) = \nu(\{0\}) + \nu(D_\beta) = \nu(\{0\})$. By definition we have $T_\beta^{-1}(Y) = U_\beta^{-1}(Y)$ for all Borel measurable subsets $Y \subseteq (0,1)$. Hence, we have $\nu \in \mathcal{M}(I, T_\beta)$. If on the other hand $\nu(\{1\}) > 0$, since $T_\beta^{-1}(1) = \emptyset$ by (i), we have $\nu \notin \mathcal{M}(I, T_\beta)$.

(iv) and (v) Recall that 1 is a periodic point of U_β if and only if β is a simple beta-number (see Remark 3.11). Then the first part of (iv) and the first part of (v) follow immediately from (ii).

Fix an arbitrary $\mu \in \mathcal{M}(I, U_\beta)$. If β is not a simple beta-number, then 1 is not a periodic point of U_β (see Remark 3.11) and it is straightforward to check that $\mu(\{1\}) = 0$. By (iii), $\mu \in \mathcal{M}(I, T_\beta)$. The second part of (iv) follows.

Assume that β is a simple beta-number. Since $\mathcal{O}_\beta^*(1)$ is a periodic orbit of U_β, then it is easy to see that $\mu(\{x\}) = \mu(\{y\})$ for all $x, y \in \mathcal{O}_\beta^*(1)$. Write $t \coloneqq \mu\bigl(\mathcal{O}_\beta^*(1)\bigr)$. When $t = 1$, we have $\mu(\{x\}) = 1/\operatorname{card} \mathcal{O}_\beta^*(1)$ for all $x \in \mathcal{O}_\beta^*(1)$. In this case, $\mu = \mu_{\mathcal{O}_\beta^*(1)}$. When $t \in [0,1)$, let us write $\nu \coloneqq \frac{1}{1-t}\bigl(\mu - t\mu_{\mathcal{O}_\beta^*(1)}\bigr)$. Then $\nu \in \mathcal{M}(I, U_\beta)$ and $\nu(\{1\}) = 0$. By (iii), $\nu \in \mathcal{M}(I, T_\beta)$. In this case, $\mu = t\mu_{\mathcal{O}_\beta^*(1)} + (1-t)\nu$. The second part of (v) follows.

(vi) Fix arbitrary $x, y \in I$ with $0 < y - x < 1/(2\beta)$. If there exists an integer i such that $i/\beta \leqslant x < y < (i+1)/\beta$, then $T_\beta(y) - T_\beta(x) = \beta(y-x)$. Otherwise, there exists an integer i such that $(i-1)/\beta < x < i/\beta \leqslant y < (i+1)/\beta$. Then we have $T_\beta(y) - T_\beta(x) = \beta(y-x) - 1 < -1/2 < -\beta|y-x|$. Similarly, we can prove $|U_\beta(y) - U_\beta(x)| \geqslant \beta|y-x|$. $\square$

Proof of Lemma 3.13. Let us write $\mathcal{K} \coloneqq \operatorname{supp} \mu$.

Assume that $0 \notin \mathcal{K}$ and denote $\delta_1 \coloneqq d(\mathcal{K}, 0) > 0$. By (3.2), for each $y \in D_\beta$, we obtain

$$\mu((y, y+\delta_1/\beta) \cap I) \leqslant \mu\bigl(U_\beta^{-1}(0,\delta_1)\bigr) = \mu((0,\delta_1)) = 0.$$

So $\mathcal{K} \cap (y, y+\delta_1/\beta) = \emptyset$ for each $y \in D_\beta$. Hence for each pair of $x, y \in \mathcal{K}$ with $|x-y| < \delta_1/\beta$, we have $(x,y) \cap D_\beta = \emptyset$ and $U_\beta(x) - U_\beta(y) = \beta(x-y)$. So $U_\beta|_{\mathcal{K}}$ is continuous and μ can be seen as an invariant measure for $(\mathcal{K}, U_\beta|_{\mathcal{K}})$. Therefore, $U_\beta(\mathcal{K}) = \mathcal{K}$ ([Ak93, p. 156]).

Assume that $1 \notin \mathcal{K}$ and denote $\delta_2 \coloneqq d(\mathcal{K}, 1) > 0$. By Proposition 3.12 (iii), $\mu \in \mathcal{M}(I, T_\beta)$. By (3.1), for each $y \in D_\beta$, we obtain

$$\mu((y - \delta_2/\beta, y) \cap I) \leqslant \mu\big(T_\beta^{-1}(1 - \delta_2, 1)\big) = \mu((1 - \delta_2, 1)) = 0.$$

So $\mathcal{K} \cap (y - \delta_2/\beta, y) = \emptyset$ for each $y \in D_\beta$. Hence for each pair of $x, y \in \mathcal{K}$ with $|x - y| < \delta_2/\beta$, we have $(x, y) \cap D_\beta = \emptyset$ and $T_\beta(x) - T_\beta(y) = \beta(x - y)$. So $T_\beta|_\mathcal{K}$ is continuous and μ can be seen as an invariant measure for $(\mathcal{K}, T_\beta|_\mathcal{K})$. Therefore, $T_\beta(\mathcal{K}) = \mathcal{K}$ ([Ak93, p. 156]). □

Proof of Proposition 3.14. (i) follows from [Pa60, Theorem 3] and Proposition 3.9 (viii), while (ii) is exactly [IT74, Lemma 4.4]. □

Proof of Lemma 3.16. (i) Assume that $1 < \beta' < \beta$. By Proposition 3.9 (xiii) and (3.8), we have $\pi_{\beta'}^*(1) \prec \pi_\beta^*(1)$. By (3.9), $\mathcal{S}_{\beta'} \subseteq \mathcal{S}_\beta$.

(ii) Assume that $\pi_\beta^*(1) = a_1 a_2 \dots$. For each $n \in \mathbb{N}$, put $A_n \coloneqq a_1 \dots a_n 00 \dots$. By Proposition 3.9 (xiv), we get that $\pi_\beta^*(1) = \lim_{\gamma \nearrow \beta} \pi_\gamma^*(1)$. Thus, for each $n \in \mathbb{N}$, there exists $\gamma_n \in (1, \beta)$ such that $A_n \preceq \pi_{\gamma_n}^*(1)$. Fix arbitrary $B = b_1 b_2 \dots \in \mathcal{S}_\beta$. Put $B_n \coloneqq b_1 \dots b_n 00 \dots$ for each $n \in \mathbb{N}$. By (3.9), for each $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we have $\sigma^k(B_n) \preceq A_n \preceq \pi_{\gamma_n}^*(1)$. Thus, $B_n \in \mathcal{S}_{\gamma_n}$. Note that $\lim_{n \to +\infty} B_n = B$, so $B \in \overline{\bigcup_{\gamma \in (1,\beta)} \mathcal{S}_\gamma}$. Since B is chosen arbitrarily, we obtain from (i) that $\mathcal{S}_\beta = \overline{\bigcup_{\gamma \in (1,\beta)} \mathcal{S}_\gamma}$. □

Proof of Lemma 3.19. If $\mathcal{K}$ is a non-empty compact set with $1 \notin \mathcal{K} = T_\beta(\mathcal{K})$, then the largest point in $\mathcal{K}$ is strictly smaller than 1. By Proposition 3.9 (vi), (xiv), and (3.8), there exists $\beta' \in (1, \beta)$ such that $\max\{\pi_\beta^*(x) : x \in \mathcal{K}\} \prec \pi_{\beta'}^*(1)$. Furthermore, by Proposition 3.9 (vi), (viii), and the fact that $\pi_{\beta'}^*(1) \in X_\beta$ (see (3.9) and Lemma 3.16 (i)), we have $\max\{\pi_\beta(x) : x \in \mathcal{K}\} \preceq \pi_{\beta'}^*(1)$. By Proposition 3.9 (iii), we get $\sigma(\pi_\beta(\mathcal{K})) = \pi_\beta(\mathcal{K})$. So if $z \in \mathcal{K}$ then $\sigma^n(\pi_\beta(z)) \preceq \pi_{\beta'}^*(1)$ for all $n \in \mathbb{N}_0$, and thus by (3.9), $\pi_\beta(z) \in \mathcal{S}_{\beta'}$. Hence, by the definition of H_β^γ and Proposition 3.9 (iv), we have $\mathcal{K} = h_\beta(\pi_\beta(\mathcal{K})) \subseteq h_\beta(\mathcal{S}_{\beta'}) = H_\beta^{\beta'}$. By (3.11) and Lemma 3.16 (i), we have $H_\beta^{\beta'} \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$. Hence, $\mathcal{K} \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$.

Now let $\mathcal{K}^*$ be a non-empty compact set with $1 \notin \mathcal{K}^* = U_\beta(\mathcal{K}^*)$. Applying Proposition 3.12 (i), we have $\mathcal{K}^* \cap D_\beta = \emptyset$, so $T_\beta(x) = U_\beta(x)$ for each $x \in \mathcal{K}^*$. Thus, $T_\beta(\mathcal{K}^*) = \mathcal{K}^*$. Therefore, there exists $\beta' \in (1, \beta)$ such that $\mathcal{K}^* \subseteq H_\beta^\gamma$ for each $\gamma \in (\beta', \beta)$. □

Proof of Lemma 3.20. Define $\delta \coloneqq d\big(H_\beta^\gamma, 1\big)$. By (3.8) and Proposition 3.9 (i) and (xiii), we have $\pi_\gamma^*(1) \prec \pi_\beta^*(1) \preceq \pi_\beta(1)$. Hence $\pi_\beta(1), \pi_\beta^*(1) \notin \mathcal{S}_\gamma$ (see (3.9)). So by Proposition 3.9 (x), (xi), and (3.11), we have $1 \notin H_\beta^\gamma$ and $0 < \delta \leqslant 1$.

Assume that $x, y \in H_\beta^\gamma$ satisfy $d_\beta(\pi_\beta(x), \pi_\beta(y)) = \beta^{-n}$ and $x < y$, then

$$\pi_\beta(x) = a_1 \dots a_{n-1} b_n b_{n+1} \dots, \quad \pi_\beta(y) = a_1 \dots a_{n-1} c_n c_{n+1} \dots,$$

where $b_n < c_n$. Then by Proposition 3.9 (iv) and the definition of h_β,

$$d(x, y) = d(h_\beta(\pi_\beta(x)), h_\beta(\pi_\beta(y))) = \beta^{-n+1} d(h_\beta(b_n b_{n+1} \dots), h_\beta(c_n c_{n+1} \dots)) \leqslant \beta^{-n+1}. \tag{A.3}$$

Moreover, by the definition of h_β and H_β^γ, we have

$$h_\beta(b_n b_{n+1} \dots) = (b_n + h_\beta(b_{n+1} \dots))/\beta \leqslant (b_n + 1 - \delta)/\beta \quad \text{and}$$
$$h_\beta(c_n c_{n+1} \dots) = (c_n + h_\beta(c_{n+1} \dots))/\beta \geqslant c_n/\beta \geqslant (b_n + 1)/\beta.$$

So we have $d(h_\beta(b_n b_{n+1} \dots), h_\beta(c_n c_{n+1} \dots)) \geqslant \delta/\beta$. Thus, by (A.3), we have

$$d(x, y) \geqslant \delta \beta^{-n} = \delta d_\beta(\pi_\beta(x), \pi_\beta(y)). \tag{A.4}$$

Let us write $C \coloneqq \max\{\beta, 1/\delta\} \geqslant 1$. Combining (A.3) and (A.4), we have

$$C^{-1} d(x, y) \leqslant d_\beta(\pi_\beta(x), \pi_\beta(y)) \leqslant C d(x, y).$$ □

Proof of Lemma 3.21. By (3.9), $\mathcal{S}_\gamma$ is closed and $\sigma(\mathcal{S}_\gamma) \subseteq \mathcal{S}_\gamma$. By Proposition 3.9 (iv), (3.11), and Lemma 3.20, $\pi_\beta|_{H_\beta^\gamma}$ is bi-Lipschitz with inverse $h_\beta|_{\mathcal{S}_\gamma}$ and H_β^γ is closed. By Proposition 3.9 (iii), $T_\beta\bigl(H_\beta^\gamma\bigr) \subseteq H_\beta^\gamma$. Since $T_\beta|_{H_\beta^\gamma} = h_\beta|_{\mathcal{S}_\gamma} \circ \sigma|_{\mathcal{S}_\gamma} \circ \pi_\beta|_{H_\beta^\gamma}$ (see Proposition 3.9 (iii) and (v)), we obtain (i).

(ii) follows from Proposition 3.12 (vi).

To verify (iii), assume that γ is a simple beta-number. Then $(\mathcal{S}_\gamma, \sigma)$ is a subshift of finite type (see e.g. [Bl89, Proposition 4.1]), and therefore $\sigma|_{\mathcal{S}_\gamma}$ is open (see e.g. [URM22, Theorem 3.2.12]). By Lemma 3.20 and Proposition 3.9 (iv), we know that $\pi_\beta|_{H_\beta^\gamma}$ and $h_\beta|_{\mathcal{S}_\gamma}$ are homeomorphisms. It follows that $T_\beta|_{H_\beta^\gamma} = h_\beta|_{\mathcal{S}_\gamma} \circ \sigma|_{\mathcal{S}_\gamma} \circ \pi_\beta|_{H_\beta^\gamma}$ is open. □

Proof of Proposition 3.23. Denote $\pi_\beta^*(1) = a_1 a_2 \dots$ in this proof. Recall that $(\pi_\beta \circ T_\beta)(x) = (\sigma \circ \pi_\beta)(x)$ and $(h_\beta \circ \pi_\beta)(x) = x$ for each $x \in I$ (see Proposition 3.9 (iii) and (iv)). By (3.13) and Proposition 3.9 (vi), $x \in I^n$ if and only if

$$\varepsilon_1 \varepsilon_2 \dots \varepsilon_n (0)^\infty \preceq \pi_\beta(x) \prec \varepsilon_1 \varepsilon_2 \dots (\varepsilon_n + 1)(0)^\infty. \tag{A.5}$$

(i) By (3.13) and Proposition 3.9 (vi), $I_1^n \cap I_2^n = \emptyset$ for each I_1^n, $I_2^n \in W^n$ with $I_1^n \neq I_2^n$. For each $x \in [0,1)$, assume $\pi_\beta(x) = x_1 x_2 \dots x_n \dots$. Then $x \in I(x_1, \dots, x_n)$. So $[0,1) = \bigcup_{J^n \in W^n} J^n$.

(ii) For arbitrary x, $y \in I^n$, assume that $\pi_\beta(x) = \varepsilon_1 \dots \varepsilon_n x_1 x_2 \dots$ and $\pi_\beta(y) = \varepsilon_1 \dots \varepsilon_n y_1 y_2 \dots$. So $y - x = h_\beta(\pi_\beta(y)) - h_\beta(\pi_\beta(x)) = \beta^{-n} \sum_{i=1}^{+\infty} \beta^{-i}(y_i - x_i)$ and

$$T_\beta^m(y) - T_\beta^m(x) = h_\beta(\sigma^m(\pi_\beta(y))) - h_\beta(\sigma^m(\pi_\beta(x))) = \beta^{m-n} \sum_{i=1}^{+\infty} \beta^{-i}(y_i - x_i) = \beta^m (y - x),$$

as required.

(iii) Fix arbitrary $b \in \{0, \dots, \varepsilon_n - 1\}$. Since $(\varepsilon_1, \dots, \varepsilon_n)$ is admissible, by Proposition 3.14 (i), there exist y_b, z_b with $\pi_\beta(y_b) = \varepsilon_1 \varepsilon_2 \dots \varepsilon_{n-1} b (0)^\infty$ and $\pi_\beta(z_b) = \varepsilon_1 \varepsilon_2 \dots \varepsilon_{n-1}(b+1)(0)^\infty$. Since π_β is strictly increasing (see Proposition 3.9 (vi)), using (A.5), $x \in I(\varepsilon_1, \dots, \varepsilon_{n-1}, b)$ if and only if $y_b \leqslant x < z_b$. Now using (3.7), $\operatorname{diam} I(\varepsilon_1, \dots, \varepsilon_{n-1}, b) = z_b - y_b = h_\beta(\pi_\beta(z_b)) - h_\beta(\pi_\beta(y_b)) = \beta^{-n}$. Moreover, $T_\beta^n(I(\varepsilon_1, \dots, \varepsilon_{n-1}, b)) = [0,1)$ by (ii). Therefore $I(\varepsilon_1, \dots, \varepsilon_{n-1}, b) \in W_0^n$ with the right endpoint $z_b = h_\beta(\varepsilon_1 \dots \varepsilon_{n-1}(b+1)(0)^\infty) = (b+1)\beta^{-n} + \sum_{i=1}^{n-1} \varepsilon_i \beta^{-i}$.

(iv) By (ii), $T_\beta^n|_{I^n}$ is continuous and increasing. Write $I^n = [x, y)$ and assume $y < 1$. As $T_\beta(I^n) = [0,1)$, so $T_\beta(x) = 0 \leqslant x$ and $\lim_{z \nearrow y} T_\beta(z) = 1 > y$. By the intermediate value theorem, T_β^n has a fixed point in I^n.

(v) Denote

$$m \coloneqq \max(\{j \in \mathbb{N} : \varepsilon_{n-j+i} = a_i \text{ for all } 1 \leqslant i \leqslant j\} \cup \{0\}).$$

Let y be the left endpoint of I^n and $z \coloneqq h_\beta(A)$ with

$$A \coloneqq \varepsilon_1 \dots \varepsilon_{n-m} \pi_\beta^*(1) = \varepsilon_1 \dots \varepsilon_n a_{m+1} a_{m+2} \dots. \tag{A.6}$$

We first check that $A \in \mathcal{S}_\beta$. Fix $k \in \mathbb{N}_0$ arbitrarily. If $k < n - m$, then $\varepsilon_{k+1} \dots \varepsilon_n (0)^\infty \prec a_1 \dots a_{n-k}(0)^\infty$ by the maximality of m and Proposition 3.9 (vi). So $\sigma^k(A) \prec \pi_\beta^*(1)$. If $k \geqslant n - m$, then $\sigma^k(A) = \sigma^{k-(n-m)}\bigl(\pi_\beta^*(1)\bigr) \preceq \pi_\beta^*(1)$ by (A.6) and Proposition 3.9 (vi). We obtain $A \in \mathcal{S}_\beta$ by (3.9). Since $h_\beta(A) = z$ and $A = \varepsilon_1 \dots \varepsilon_{n-m} \pi_\beta^*(1)$, by Proposition 3.9 (ii) and (xi), $A = \pi_\beta^*(z)$.

Consider arbitrary $x \in I^n$, then $\pi_\beta(x) \in \mathcal{S}_\beta$ by Definition 3.6. So $\sigma^{n-m}(\pi_\beta(x)) \preceq \pi_\beta^*(1) = \sigma^{n-m}(A)$ by Proposition 3.14 (ii) and (A.6). Combining this with the fact that the first $n - m$ terms of $\pi_\beta(x)$ and A coincide (see (A.5)), we get $\pi_\beta(x) \preceq A$. Since h_β is non-decreasing (see Proposition 3.9 (x)), $x \leqslant z = h_\beta(A)$ for all $x \in I^n$ by Proposition 3.9 (iv). Consider arbitrary $w \in [y, z)$. By Proposition 3.9 (vii) and (ii), $\pi_\beta(w) \preceq \pi_\beta^*(z) = A \preceq \pi_\beta(z)$. Hence $w \in I^n$ for all $w \in [y, z)$ by (A.5) and (A.6).

By our discussion above, z is the right endpoint of I^n. Moreover, by (ii) and (3.14), $T_\beta^n(I^n)$ is a left closed and right open interval with left endpoint 0 and right endpoint $\beta^n(z - y) = h_\beta\bigl(\sigma^m\bigl(\pi_\beta^*(1)\bigr)\bigr) = U_\beta^m(1)$ (see Proposition 3.9 (iii) and (iv) for the last equality). □

Proof of Lemma 3.24. By Proposition 3.23 (ii),

$$|S_n\phi(x)-S_n\phi(y)| \leqslant |\phi|_\alpha \sum_{i=0}^{n-1} \big|T_\beta^i(x)-T_\beta^i(y)\big|^\alpha = |\phi|_\alpha \sum_{i=0}^{n-1} \frac{|T_\beta^n(x)-T_\beta^n(y)|^\alpha}{\beta^{(n-i)\alpha}} \leqslant \frac{|\phi|_\alpha}{\beta^\alpha-1}\big|T_\beta^n(x)-T_\beta^n(y)\big|^\alpha. \qquad \square$$

Proof of Lemma 3.26. (i) Fix arbitrary $x\in I$. By (3.15), Proposition 3.9 (ii), and Lemma 3.8, $x\in Z_\beta$ if and only if there exists $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{N}$ with $z_n>0$ such that

$$\pi_\beta(x)=z_1\dots z_n(0)^\infty.$$

By Proposition 3.9 (iv) and (v), $T_\beta^n(x)=\big(T_\beta^n\circ h_\beta\circ\pi_\beta\big)(x)=(h_\beta\circ\sigma^n\circ\pi_\beta)(x)$. So the condition above is equivalent to the condition that there exists $n\in\mathbb{N}$ such that $x\in T_\beta^{-n}(0)\smallsetminus T_\beta^{-(n-1)}(0)$. Hence

$$Z_\beta=\bigcup_{n\in\mathbb{N}}\big(T_\beta^{-n}(0)\smallsetminus T_\beta^{-(n-1)}(0)\big)=\Big(\bigcup_{n\in\mathbb{N}}T_\beta^{-n}(0)\Big)\smallsetminus\{0\}.$$

Similarly, we can prove that $Z_\beta=\bigcup_{n\in\mathbb{N}}U_\beta^{-n}(1)$. In particular, by Proposition 3.12 (i), $D_\beta=U_\beta^{-1}(1)\subseteq Z_\beta$.

(ii) Fix arbitrary $W\subseteq I$. By Proposition 3.9 (xi), we have $h_\beta^{-1}(W)=\pi_\beta(W)\cup\pi_\beta^*(W)$. By (3.15),

$$\begin{aligned} h_\beta^{-1}(W)&=\pi_\beta^*(W)\cup\pi_\beta(W)=\pi_\beta^*(W)\cup(\pi_\beta(W\smallsetminus Z_\beta)\cup\pi_\beta(W\cap Z_\beta))\\ &=\big(\pi_\beta^*(W)\cup\pi_\beta^*(W\smallsetminus Z_\beta)\big)\cup\pi_\beta(W\cap Z_\beta)=\pi_\beta^*(W)\cup\pi_\beta(W\cap Z_\beta).\end{aligned}$$

For each $A\in\pi_\beta(Z_\beta)$, A has finitely many nonzero terms and $(0)^\infty\notin\pi_\beta(Z_\beta)$ (see Proposition 3.9 (ii) and Lemma 3.8). By Proposition 3.9 (i), (ii), and Lemma 3.8, $\pi_\beta^*(x)$ has infinitely many nonzero terms for all $x\in(0,1]$ and $\pi_\beta^*(0)=(0)^\infty$. The second part of (ii) follows.

(iii) By Proposition 3.9 (iv) and (v), for each $x\in I$,

$$\begin{aligned} T_\beta^n(x)&=\big(T_\beta^n\circ h_\beta\circ\pi_\beta\big)(x)=(h_\beta\circ\sigma^n\circ\pi_\beta)(x) \quad \text{and}\\ U_\beta^n(x)&=\big(U_\beta^n\circ h_\beta\circ\pi_\beta^*\big)(x)=\big(h_\beta\circ\sigma^n\circ\pi_\beta^*\big)(x),\end{aligned}$$

so these, together with (3.15), give (iii).

(iv) Assume that $x\in(0,1]\smallsetminus Z_\beta$, so that $\pi_\beta(x)=\pi_\beta^*(x)$ by (3.15). From the fact that π_β^* is left-continuous, the fact that $\pi_\beta^*(y)\preceq\pi_\beta(y)$ for all $y\in I$, and the fact that π_β is strictly increasing (see Proposition 3.9 (ix), (ii), and (vi)), we have

$$\pi_\beta^*(x)=\lim_{y\nearrow x}\pi_\beta^*(y)\preceq\lim_{y\nearrow x}\pi_\beta(y)\preceq\pi_\beta(x).$$

So $\lim_{y\nearrow x}\pi_\beta(y)=\pi_\beta(x)$. Combining this with the fact that π_β is right-continuous on $[0,1)$, and that $x\in(0,1]\smallsetminus Z_\beta$ was arbitrary, we see that π_β is continuous on $I\smallsetminus Z_\beta$. The fact that π_β^* is continuous on $I\smallsetminus Z_\beta$ can be proved similarly.

(v) If $x\in Z_\beta$ then $(0)^\infty\in\mathcal{O}^\sigma(\pi_\beta(x))$ by Proposition 3.9 (ii) and Lemma 3.8. So

$$\pi_\beta(Z_\beta)\subseteq\Bigg(\bigcup_{n=1}^{+\infty}\sigma^{-n}((0)^\infty)\Bigg)\smallsetminus\{(0)^\infty\}.$$

If $\mu\in\mathcal{M}(X_\beta,\sigma)$ and $n\in\mathbb{N}$, then $\mu(\sigma^{-n}((0)^\infty))=\mu(\{(0)^\infty\})$ and $(0)^\infty\in\sigma^{-n}((0)^\infty)$. This implies that $\mu\big(\big(\bigcup_{n=1}^{+\infty}\sigma^{-n}((0)^\infty)\big)\smallsetminus\{(0)^\infty\}\big)=0$, and therefore $\mu(\pi_\beta(Z_\beta))=0$. $\square$

Proof of Proposition 3.27. (i) By Proposition 3.9 (iii), $\sigma\big(\pi_\beta^*(I)\big)\subseteq\pi_\beta^*(I)$. By Lemma 3.26 (ii) applied to $W=I$, we have $X_\beta=\pi_\beta^*(I)\cup\pi_\beta(Z_\beta)$. Thus, by Lemma 3.26 (v), we have that $\mathcal{M}\big(\pi_\beta^*(I),\sigma\big)$ can be naturally identified with $\mathcal{M}(X_\beta,\sigma)$. More precisely, $\mu(\,\cdot\,)\mapsto\mu\big(\,\cdot\,\cap\pi_\beta^*(I)\big)$ is a bijection from

$\mathcal{M}\big(\pi_\beta^*(I),\sigma\big)$ to $\mathcal{M}(X_\beta,\sigma)$. So H_β can be seen as the pushforward of $h_\beta|_{\pi_\beta^*(I)}$ from $\mathcal{M}\big(\pi_\beta^*(I),\sigma\big)$ to $\mathcal{P}(I)$. Proposition 3.9 (v) implies that $H_\beta(\mathcal{M}(X_\beta,\sigma))\subseteq\mathcal{M}(I,U_\beta)$.

For each $\mu\in\mathcal{M}(I,U_\beta)$ and each Borel measurable subset $Y\subseteq X_\beta$, by (3.16) and Proposition 3.9 (iv), we have

$$G_\beta(\mu)(Y)=\mu\big(\big(\pi_\beta^*\big)^{-1}(Y)\big)=\mu\big(\big(\pi_\beta^*\big)^{-1}(Y\cap\pi_\beta^*(I))\big)=\mu\big(h_\beta\big(Y\cap\pi_\beta^*(I)\big)\big). \tag{A.7}$$

Hence we derive that $(H_\beta\circ G_\beta)(\mu)=\mu$ for all $\mu\in\mathcal{M}(I,U_\beta)$. More precisely, for each Borel measurable subset $W\subseteq I$, by (3.17), (A.7), Lemma 3.26 (ii), and Proposition 3.9 (iv),

$$\begin{aligned}(H_\beta\circ G_\beta)(\mu)(W)&=G_\beta(\mu)\big(h_\beta^{-1}(W)\big)\\&=\mu\big(h_\beta\big(h_\beta^{-1}(W)\cap\pi_\beta^*(I)\big)\big)\\&=\mu\big(h_\beta\big(\big(\pi_\beta^*(W)\cup\pi_\beta(W\cap Z_\beta)\big)\cap\pi_\beta^*(I)\big)\big)\\&=\mu\big(h_\beta\big(\big(\pi_\beta^*(W)\cap\pi_\beta^*(I)\big)\cup\big(\pi_\beta(W\cap Z_\beta)\cap\pi_\beta^*(I)\big)\big)\big)\\&=\mu\big(h_\beta\big(\pi_\beta^*(W)\big)\big)\\&=\mu(W).\end{aligned}$$

For each $\nu\in\mathcal{M}(X_\beta,\sigma)$ and each Borel measurable subset $W\subseteq I$, by (3.17) and Lemma 3.26 (ii) and (v), we have

$$H_\beta(\nu)(W)=\nu\big(h_\beta^{-1}(W)\big)=\nu\big(\pi_\beta^*(W)\cup\pi_\beta(W\cap Z_\beta)\big)=\nu\big(\pi_\beta^*(W)\big). \tag{A.8}$$

Hence we derive that $(G_\beta\circ H_\beta)(\nu)=\nu$ for all $\nu\in\mathcal{M}(X_\beta,\sigma)$. More precisely, for each Borel measurable subset $Y\subseteq X_\beta$, by (3.16) and (A.8),

$$(G_\beta\circ H_\beta)(\nu)(Y)=H_\beta(\nu)\big(\big(\pi_\beta^*\big)^{-1}(Y)\big)=\nu\big(\pi_\beta^*\big(\big(\pi_\beta^*\big)^{-1}(Y)\big)\big)=\nu(Y).$$

By the above, H_β is a bijection from $\mathcal{M}(X_\beta,\sigma)$ to $\mathcal{M}(I,U_\beta)$. Moreover, by Proposition 3.9 (x), H_β is continuous from $\mathcal{M}(X_\beta,\sigma)$ to $\mathcal{P}(I)$.

The weak* compactness of $\mathcal{M}(X_\beta,\sigma)$ follows immediately from the compactness of X_β and the continuity of σ. By [Wal82, Theorem 6.4], the set of probability measures on I is Hausdorff in the weak* topology, hence $\mathcal{M}(I,U_\beta)$ is Hausdorff, and therefore H_β is a homeomorphism from $\mathcal{M}(X_\beta,\sigma)$ to $\mathcal{M}(I,U_\beta)$, with $G_\beta^{-1}=H_\beta$.

(ii) follows immediately from (i), and the weak* compactness of $\mathcal{M}(X_\beta,\sigma)$.

(iii) Since G_β is the pushforward of π_β^*, for each $\mu\in\mathcal{M}(X_\beta,\sigma)$, by Proposition 3.9 (iv) and statement (i), we have

$$\int_I\phi\,\mathrm{d}H_\beta(\mu)=\int_I\big(\phi\circ h_\beta\circ\pi_\beta^*\big)\,\mathrm{d}H_\beta(\mu)=\int_{X_\beta}(\phi\circ h_\beta)\,\mathrm{d}(G_\beta\circ H_\beta)\mu=\int_{X_\beta}(\phi\circ h_\beta)\,\mathrm{d}\mu.$$

By (i), we obtain the required identities

$$Q(U_\beta,\phi)=Q\big(\sigma|_{X_\beta},\phi\circ h_\beta\big)\quad\text{ and }\quad\mathcal{M}_{\max}(U_\beta,\phi)=H_\beta\big(\mathcal{M}_{\max}\big(\sigma|_{X_\beta},\phi\circ h_\beta\big)\big).$$

Since $\mathcal{M}(X_\beta,\sigma)$ is weak* compact and h_β is continuous (see Proposition 3.9 (x)), $\mathcal{M}_{\max}(\sigma|_{X_\beta},\phi\circ h_\beta)$ is non-empty.

(iv) First note that $\pi_\beta(0)=\pi_\beta^*(0)=(0)^\infty$ (see Lemma 3.8). Next, by Proposition 3.9 (ii), any $\underline{a}\in\pi_\beta(I)\smallsetminus\pi_\beta^*(I)$ is of the form $\underline{a}=a_1a_2\dots a_n(0)^\infty$ with $a_n>0$, and therefore not a σ-periodic point. Applying Lemma 3.26 (ii) in the case $W=I$ gives

$$X_\beta=h_\beta^{-1}(I)\subseteq\pi_\beta(I)\cup\pi_\beta^*(I). \tag{A.9}$$

Any (X_β,σ)-periodic orbit $\mathcal{O}$ cannot intersect $\pi_\beta(I)\smallsetminus\pi_\beta^*(I)$, as noted above, therefore (A.9) implies that $\mathcal{O}\subseteq\pi_\beta^*(I)$. So the fact that $h_\beta\circ\sigma=U_\beta\circ h_\beta$ on $\pi_\beta^*(I)$ (by Proposition 3.9 (v)) implies that $h_\beta(\mathcal{O})$ is an (I,U_β)-orbit and that $U_\beta(h_\beta(\mathcal{O}))=h_\beta(\mathcal{O})$. So $h_\beta(\mathcal{O})$ is an (I,U_β)-periodic orbit.

To prove that $\operatorname{card} \mathcal{O} = \operatorname{card} h_\beta(\mathcal{O})$ it suffices to show that h_β is injective on $\mathcal{O}$. Indeed, if h_β were not injective on $\mathcal{O}$, then by Proposition 3.9 (ii) and (xi), there would exist a point in $\mathcal{O}$ of the form $z_1 z_2 \dots z_n (0)^\infty$ for $z_n > 0$, and this is not a σ-periodic point, a contradiction. □

Proof of Proposition 3.28. If β is not a simple beta-number, then $\mathcal{M}(I, T_\beta) = \mathcal{M}(I, U_\beta)$ is weak* compact (see Proposition 3.12 (iv) and Proposition 3.27 (ii)). So (i) holds and both (ii) and (iii) follow immediately from (i).

If β is a simple beta-number, by Proposition 3.12 (v), it suffices to prove that $\mu_{\mathcal{O}^*_\beta(1)}$ is contained in the weak* closure of $\mathcal{M}(I, T_\beta)$. Note that $\mathcal{O}^*_\beta(1)$ and $\mathcal{O}^\sigma\big(\pi^*_\beta(1)\big)$ are periodic orbits of U_β and σ, respectively. By [Si76, p. 249], the periodic measures are weak* dense in $\mathcal{M}(X_\beta, \sigma)$, so there exists a sequence of periodic orbits $\{\mathcal{O}_n\}$ of (X_β, σ) satisfying:

(a) $\mathcal{O}_n \neq \mathcal{O}^\sigma(\pi^*_\beta(1))$ for all $n \in \mathbb{N}$, and

(b) $\mu_{\mathcal{O}_n}$ converges to $\mu_{\mathcal{O}^\sigma(\pi^*_\beta(1))}$ in the weak* topology as n tends to $+\infty$.

By Proposition 3.27 (iv), $\{h_\beta(\mathcal{O}_n)\}_{n\in\mathbb{N}}$ are U_β-periodic orbits. Note that $h_\beta\big(\mathcal{O}^\sigma\big(\pi^*_\beta(1)\big)\big) = \mathcal{O}^*_\beta(1)$, so that by Proposition 3.27 (i), $H_\beta(\mu_{\mathcal{O}_n}) = \mu_{h_\beta(\mathcal{O}_n)}$ converges to $H_\beta\big(\mu_{\mathcal{O}^\sigma(\pi^*_\beta(1))}\big) = \mu_{\mathcal{O}^*_\beta(1)}$ in the weak* topology, and $h_\beta(\mathcal{O}_n) \neq \mathcal{O}^*_\beta(1)$ for each $n \in \mathbb{N}$. But $H_\beta(\mu_{\mathcal{O}_n}) \in \mathcal{M}(I, T_\beta)$ for each $n \in \mathbb{N}$, by Proposition 3.12 (iii), so (i) follows. Both (ii) and (iii) follow immediately from (i). □

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