JOINT TYPICAL PERIODIC OPTIMIZATION: SYSTEMS WITH STABLE HYPERBOLICITY

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Abstract. The framework of joint typical periodic optimization, in which both the dynamical system and the potential function are allowed to vary simultaneously, was introduced in [HHJL25], in a direction motivated by the work of Yang, Hunt & Ott [YHO00]. For certain classes of hyperbolic systems, it was shown there that optimizing periodic orbits persist under simultaneous perturbation, yielding joint locking sets that contain open dense subsets of the relevant product spaces. In the present article we broaden the scope of this theory, by developing an axiomatic joint perturbation framework that accommodates a wider class of stably hyperbolic systems, and by establishing new joint typical periodic optimization results for several natural and important families: Axiom A diffeomorphisms with the no-cycle property, hyperbolic rational maps on the Riemann sphere, real quadratic polynomials, and C^r maps in one dimension.

Contents

1. Introduction

1.1. **Background.** Ergodic optimization is the study of those probability measures, invariant under some dynamical system, that optimize the space average of a given potential function. For (X, d) a compact metric space, and $f\colon X \to X$ a continuous self-map, let $\mathcal{M}(X, f)$ denote the space of *f-invariant Borel probability measures.* For a continuous function $\phi\colon X \to \mathbb{R}$, the corresponding *maximum ergodic average* is defined as

$$Q(X, f, \phi) := \sup\left\{ \int \phi \,\mathrm{d}\mu : \mu \in \mathcal{M}(X, f) \right\}. \tag{1.1}$$

A measure $\mu \in \mathcal{M}(X, f)$ that attains the maximum ergodic average (1.1) is called *(f, ϕ)-maximizing*, and the set of (f, ϕ)-maximizing measures is denoted by $\mathcal{M}_{\max}(X, f, \phi)$, in other words

$$\mathcal{M}_{\max}(X, f, \phi) := \left\{ \mu \in \mathcal{M}(X, f) : \int \phi \,\mathrm{d}\mu = Q(X, f, \phi) \right\}. \tag{1.2}$$

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For simplicity we write $Q(f,\phi)$ and $\mathcal{M}_{\max}(f,\phi)$ whenever there is no possibility of confusion regarding the space X.

We say that (f,ϕ) has the *periodic optimization property* (or *PO property*) if there exists a unique (f,ϕ)-maximizing measure, and this measure is supported on a single f-periodic orbit. If $\mathcal{P}$ is a Banach space consisting of continuous real-valued functions defined on X, we say that $(f,\mathcal{P})$ has the *typical periodic optimization property* (or *TPO property*) if there exists an open dense subset $V(f)$ of $\mathcal{P}$ such that for every $\phi \in V(f)$, the pair (f,ϕ) has the PO property. For $\phi \in \mathcal{P}$, we say that (f,ϕ) has the *locking property* if (f,ϕ) has the PO property and $\mathcal{M}_{\max}(f,\psi) = \mathcal{M}_{\max}(f,\phi)$ for all $\psi \in \mathcal{P}$ sufficiently close to ϕ. We write

$$\mathrm{Lock}(f,\mathcal{P}) \coloneqq \{\phi \in \mathcal{P} : (f,\phi) \text{ has the locking property}\},$$

and call $\mathrm{Lock}(f,\mathcal{P})$ the *locking set* of $(f,\mathcal{P})$.

If $\mathcal{F}$ is a topological space consisting of self-maps of X, and $\mathcal{F}\times\mathcal{P}$ is equipped with the product topology, we say that $\mathcal{F}\times\mathcal{P}$ has the *joint typical periodic optimization property* (or *Joint TPO property*) if there exists an open dense subset U of $\mathcal{F}\times\mathcal{P}$ such that every pair $(f,\phi)\in U$ has the PO property. We write

$$\mathrm{Lock}(\mathcal{F},\mathcal{P}) \coloneqq \{(f,\phi) \in \mathcal{F}\times\mathcal{P} : (f,\phi) \text{ has the locking property}\} \tag{1.3}$$

and refer to $\mathrm{Lock}(\mathcal{F},\mathcal{P})$ as the *joint locking set* of $(\mathcal{F},\mathcal{P})$.

The field of ergodic optimization was significantly influenced by a 1999 conjecture of Yuan & Hunt [YH99], that $(f,\mathcal{P})$ has the TPO property whenever f is uniformly hyperbolic (an expanding map or an Axiom A diffeomorphism), and $\mathcal{P}$ is the space of Lipschitz functions. After important work towards this conjecture (see [Bou01, Bou08, BQ07, CLT01, Mo08, QS12]), it was eventually resolved by Contreras [Co16] in the case of expanding maps, and by Huang, Lian, Ma, Xu & Zhang [HLMXZ25] for Axiom A diffeomorphisms. Going beyond uniform hyperbolicity, Li & Zhang [LZ25] established TPO in the complex dynamics setting of expanding Thurston maps. We refer the reader to surveys [Boc18, Je06, Je19] for more detailed discussion on ergodic optimization.

In a direction suggested by the work of Yang, Hunt & Ott [YHO00], and by perspectives highlighted in the survey [Je19], the notion of joint typical periodic optimization was introduced in [HHJL25], where, for $\mathcal{P}$ any Banach space of α-Hölder functions, Joint TPO was established for $\mathcal{F}$ taken to be each of (i) the class of open distance-expanding Lipschitz maps on a locally connected space, (ii) the class of Anosov diffeomorphisms on a compact smooth manifold equipped with a Riemannian metric, and (iii) the class of beta-transformations on the unit interval.

The present article builds on the joint perturbation framework of [HHJL25], extending the scope of Joint TPO to various families $\mathcal{F}$ of maps enjoying some form of hyperbolicity. A central theme is that Joint TPO can be established for maps satisfying an appropriately abstract form of stable hyperbolicity, via a broad axiomatic joint perturbation framework, allowing us to deduce Joint TPO for several notable specific classes of hyperbolic dynamical systems.

This axiomatic strategy consists of the following four steps:

(1) Formulate abstract hyperbolic assumptions ensuring some structural stability of invariant sets, uniform hyperbolicity, and a Mañé cohomology lemma with uniform seminorm control.

(2) Prove a joint perturbation theorem showing that if a periodic orbit is maximizing for (f,ϕ), then nearby pairs (g,ψ) admit nearby periodic maximizing orbits.

(3) Articulate a sufficient condition for a map-function pair to lie in the interior of the joint locking set.

(4) Verify that joint typical periodic optimization does indeed hold for the various specific classes of hyperbolic systems considered.

In particular, by proceeding along these lines, we succeed in extending the joint typical periodic optimization theory of [HHJL25] to (i) the class of Axiom A diffeomorphisms with the no-cycle property, (ii) the class of hyperbolic rational maps on the Riemann sphere, (iii) the class of real quadratic polynomials on the Riemann sphere, (iv) certain classes of real one-dimensional smooth systems. and (v) the logistic family on $[0, 1]$.

1.2. **Main results.** Throughout the article, we denote by M a compact smooth manifold (without boundary), with Riemannian metric $|\cdot|$ on M, and d the induced distance function.

For $\alpha \in (0, 1]$, let $C^{0,\alpha}(M, \mathbb{R})$ denote the Banach space of real-valued α-Hölder functions on M. Let $C^1(M, \mathbb{R})$ denote the Banach space of real-valued C^1 functions on M, equipped with the norm $\|\phi\|_{C^1,M} \coloneqq \|\phi\|_{\infty,M} + \|\mathrm{D}\phi\|_{\infty,M}$, where $\mathrm{D}\phi$ is the derivative of ϕ.

Recall that a diffeomorphism $f : M \to M$ is *Axiom A* if its nonwandering set $\Omega(f)$ is uniformly hyperbolic (admitting a $\mathrm{D}f$-invariant splitting of the tangent bundle over $\Omega(f)$ into uniformly contracting stable and uniformly expanding unstable subbundles) and has dense periodic points. By Smale's spectral decomposition theorem [Sm67], $\Omega(f)$ decomposes uniquely into finitely many disjoint compact transitive pieces, the *basic sets*, and f satisfies the *no-cycle condition* if there is no cyclic chain, via intersections of stable and unstable manifolds, among the basic sets. The set of Axiom A diffeomorphisms with the no-cycle condition is precisely the set of diffeomorphisms satisfying Ω-stability (see [Pa70, Pa87, Sm70]).

For each $r \in \mathbb{N}$, let $\mathrm{Diff}^r(M)$ denote the space of C^r diffeomorphisms on M, equipped with the C^r topology, and define $\mathcal{A}^r(M)$ to be the subspace of $\mathrm{Diff}^r(M)$ consisting of those Axiom A diffeomorphisms with the no-cycle property. We prove:

Theorem A (Joint TPO for Axiom A diffeomorphisms with no cycles). *Let $r \in \mathbb{N}$, $\alpha \in (0, 1]$, and $\mathcal{P}$ denote either $C^1(M, \mathbb{R})$ or $C^{0,\alpha}(M, \mathbb{R})$. Then $\mathcal{A}^r(M) \times \mathcal{P}$ has the Joint TPO property. Moreover, the joint locking set $\mathrm{Lock}(\mathcal{A}^r(M), \mathcal{P})$ is itself an open dense subset of $\mathcal{A}^r(M) \times \mathcal{P}$.*

Note that for every $r \in \mathbb{N}$, the space $\mathcal{A}^r(M)$ is second countable (see e.g. [Hi76, p. 35]), so combining Theorem A and [Ke95, Lemma 8.42] gives:

Corollary 1.1. *For any $r \in \mathbb{N}$ and $\alpha \in (0, 1]$, there exists a residual subset H of $C^{0,\alpha}(M, \mathbb{R})$ (resp. $C^1(M, \mathbb{R})$) such that for each $\phi \in H$, there is an open dense subset $V(\phi)$ of $\mathcal{A}^r(M)$ such that if $f \in V(\phi)$ then (f, ϕ) has the periodic optimization property.*

Beyond the class of Axiom A diffeomorphisms with the no-cycle property, our methods can be applied to various other classes of hyperbolic dynamical systems. One such class consists of hyperbolic rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. For $m \in \mathbb{N}$, let $\mathcal{R}^m$ denote the collection of rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree m, equipped with the standard topology on parameter space (which is embedded homeomorphically as the complement of an algebraic hypersurface in $(2m+1)$-dimensional complex projective space $\mathbb{CP}^{2m+1}$, cf. e.g. [Be91, p. 47], [Mi93, Appendix B], [Se79]). For every $f \in \mathcal{R}^m$, let $J(f)$ denote the *Julia set* of f (cf. [Mi06, Definition 4.2]). If $m \geqslant 2$, a map $f \in \mathcal{R}^m$ is said to be *hyperbolic* if it is expanding on its Julia set, i.e., there exists a conformal metric with induced norm $|\cdot|$, and $\lambda > 1$, such that

$$|\mathrm{D}f(v)| \geqslant \lambda |v| \tag{1.4}$$

for all $z \in J(f)$ and $v \in T_z\widehat{\mathbb{C}}$ (cf. [Mi06, p. 205]). For $m \geqslant 2$, defining $\mathcal{HR}^m$ to be the collection of all hyperbolic rational maps of degree m, we prove:

Theorem B (Joint TPO for hyperbolic rational maps). *Let $m \geqslant 2$, $\alpha \in (0, 1]$, and $\mathcal{P}$ denote either $C^{0,\alpha}\big(\widehat{\mathbb{C}}, \mathbb{R}\big)$ or $C^1\big(\widehat{\mathbb{C}}, \mathbb{R}\big)$. Then $\mathcal{HR}^m \times \mathcal{P}$ has the Joint TPO property. Moreover, the joint locking set $\mathrm{Lock}(\mathcal{HR}^m, \mathcal{P})$ is itself an open dense subset of $\mathcal{HR}^m \times \mathcal{P}$.*

Note that $\mathcal{HR}^m$ is open in $\mathcal{R}^m$ (see [Mi06, p. 205]). Conditional on the Density of Hyperbolicity Conjecture (see e.g. [Ly99, p. 87], [Mc94, p. 4]), Theorem B ensures that the Joint TPO property holds on all of $\mathcal{R}^m \times C^{0,\alpha}\big(\widehat{\mathbb{C}}, \mathbb{R}\big)$, $\alpha \in (0, 1]$, and on all of $\mathcal{R}^m \times C^1\big(\widehat{\mathbb{C}}, \mathbb{R}\big)$.

Graczyk & Światek [GS97] and Lyubich [Ly97, p. 4] proved that hyperbolicity (in the sense of (1.4)) is a dense property among real quadratic polynomials. If $\mathcal{Q}$ denotes the set of real quadratic polynomials (i.e., those maps f of the form $f(z) = z^2 + c$ for $c \in [-2, 1/4]$) on $\widehat{\mathbb{C}}$, equipped with the standard topology on the parameter space $[-2, 1/4]$, we prove:

Theorem C (Joint TPO for real quadratic polynomials). *Let $\alpha \in (0, 1]$, and let $\mathcal{P}$ denote either $C^{0,\alpha}(\widehat{\mathbb{C}}, \mathbb{R})$ or $C^1(\widehat{\mathbb{C}}, \mathbb{R})$. Then $\mathcal{Q} \times \mathcal{P}$ has the Joint TPO property.*

Our methods can also be applied to various real one-dimensional dynamical systems. If M is either a compact interval or the circle, and $r \in \mathbb{N}$, we let $C^r(M, M)$ denote the space of C^r maps on M equipped with the C^r topology, and prove:

Theorem D (Joint TPO for C^r one-dimensional maps). *Let M be a compact interval or the circle, and $r \in \mathbb{N}$. If $\alpha \in (0, 1]$, and $\mathcal{P}$ denotes either $C^{0,\alpha}(M, \mathbb{R})$ or $C^1(M, \mathbb{R})$, then $C^r(M, M) \times \mathcal{P}$ has the Joint TPO property.*

Theorem C concerns real quadratic polynomials viewed as holomorphic self-maps of the Riemann sphere $\widehat{\mathbb{C}}$, with potential functions defined on all of $\widehat{\mathbb{C}}$. The same family of maps can equivalently be parametrised as the *logistic family* $g_a : [0, 1] \to [0, 1]$, $g_a(x) := a\,x(1 - x)$, $a \in [0, 4]$, which are topologically conjugate to the real quadratic polynomials as one-dimensional dynamical systems. However, the two settings lead to distinct ergodic optimization problems: in Theorem C the potential functions are Hölder (or C^1) on the Riemann sphere $\widehat{\mathbb{C}}$, whereas below they are defined only on the interval $[0, 1]$.

Theorem E (Joint TPO for the logistic family). *Let $\alpha \in (0, 1]$, $\mathcal{P}$ denote either $C^{0,\alpha}([0, 1], \mathbb{R})$ or $C^1([0, 1], \mathbb{R})$, and $\mathcal{F} := \{g_a : a \in [0, 4]\}$, where $g_a : [0, 1] \to [0, 1]$ is defined by $g_a(x) := a\,x(1 - x)$. We equip $\mathcal{F}$ with the topology induced by the parameter space $[0, 4]$. Then $\mathcal{F} \times \mathcal{P}$ has the Joint TPO property.*

Remark 1.2. In this article, as in [HHJL25], the Joint TPO property is established for spaces of maps equipped with some topology stronger than the C^0 one. It turns out that results for the C^0 topology have a rather different flavour, and will be presented elsewhere.

1.3. **Overview of the proof strategy.** The main results of Subsection 1.2 are facilitated by the abstract hyperbolic framework developed in Section 2, and the joint perturbation theorems established in that setting. For a family $\mathcal{F}$ of Lipschitz self-maps of a compact metric space (X, d), we define a notion of *$\mathcal{F}$-stable hyperbolicity* for a map $f \in \mathcal{F}$, given as Definition 2.1 (comprising the three conditions of *Intertwining Stability* (IS), *Robust Hyperbolic Estimates* (RHE), and *Mañé Lemma* (ML)); the proof of Theorems A, B, C, D and E will then involve verifying that the appropriate $\mathcal{F}$-stable hyperbolicity holds. The key result of Section 2 is the joint perturbation theorem (Theorem 2.3): if f is $\mathcal{F}$-stably hyperbolic and $\mathcal{O}$ is an f-periodic orbit with $\mathcal{M}_{\max}(f, \phi) = \{\mu_{\mathcal{O}}\}$, then for all g near f the orbit $\mathcal{O}_g = h_g(\mathcal{O})$ is the unique (g, ϕ')-maximizing orbit for every function ϕ' in a certain explicit neighbourhood of ϕ (which shrinks at a controlled rate as g approaches f). This joint perturbation theorem underpins our whole approach, and in particular leads to the Interior Condition Theorem (Theorem 2.5): if f is $\mathcal{F}$-stably hyperbolic and $(f, \phi) \in \operatorname{Lock}(\mathcal{F}, \mathcal{P})$ with ϕ nonconstant, then (f, ϕ) lies in the interior of the joint locking set $\operatorname{Lock}(\mathcal{F}, \mathcal{P})$.

To establish Theorem A, Definition 2.1 is verified by proving Lemmas 3.2, 3.3, and 3.5. Condition (IS) in Definition 2.1 follows from Ω-stability: since Axiom A with the no-cycle condition is equivalent to Ω-stability [Pa70, Sm70], the required intertwining maps h_g, i_g exist and converge to the identity. Definition 2.1 (RHE) requires a strengthening of the well-known hyperbolicity of $\Omega(f)$ for a single map (cf. [KH95, Proposition 6.4.16]): the relevant constants must be chosen uniformly over a neighbourhood of f, necessitating a careful analysis of the adapted Riemannian metric. Condition (ML) in Definition 2.1 is established by adapting the analysis of [STY24] (cf. also the closely related [Bou11]). Individual TPO for each fixed Axiom A diffeomorphism (Proposition 3.6) follows from [HLMXZ25], and Theorem 2.5 then allows joint locking to be deduced, as well as the fact that $\mathrm{Lock}(\mathcal{A}^r(M), \mathcal{P})$ is itself open.

To prove Theorem B, the verification of Definition 2.1 proceeds via tools from complex dynamics, and a key difference arises regarding condition (ML). For condition (IS) in Definition 2.1, the structural stability conjugacies under perturbation are provided by the theory of holomorphic motions [Ly83, MSS83], and for condition (RHE) in Definition 2.1, Lemma 4.3 shows that the expanding constants of $f|_{J(f)}$ can be chosen uniformly over a neighbourhood of f. For condition (ML) in Definition 2.1, unlike in the Axiom A case where a single Mañé lemma argument applies throughout, the nonwandering set of a hyperbolic rational map splits into the expanding Julia set and finitely many attracting periodic orbits: Lemma 4.5 constructs the sub-action u separately on each piece (via distance-expanding Mañé lemma techniques on $J(f)$, and an explicit orbit-average formula on the attracting orbits), and then patches these together.

To prove Theorem C, the stable hyperbolicity properties of Theorem B apply, and can be combined with the result of Graczyk–Świątek [GS97] and Lyubich [Ly97], that hyperbolicity is an open dense property in the real quadratic polynomial family.

To prove Theorem D, the most delicate and technically demanding result in this article, we start by noting that the density theorem of Kozlovski, Shen & van Strien [KSS07] means that any $g_0 \in C^r(M, M)$ can be approximated by a hyperbolic map f whose boundary values lie in the interior of the interval M. We then construct (via Lemma 5.3) an extension F of f to a strictly larger interval M_0 that preserves the new endpoints ∂M_0, and that is itself hyperbolic (Lemma 5.5 (iii)); in particular, F belongs to the space $C_0^r(M_0, M_0)$ of endpoint-preserving maps, to which the abstract framework applies. Proposition 5.1 then verifies that F is stably hyperbolic within $C_0^r(M_0, M_0)$, by showing that the restriction $F|_{K(F)}$ is distance-expanding in the adapted metric. Given any $\phi \in \mathcal{P}$, we extend it to a potential $\Phi \in \mathcal{P}_0$ on M_0 whose values at ∂M_0 are prescribed to lie strictly below $Q(f, \phi)$; Lemma 5.2 (Individual TPO for F) then produces a nearby $\Psi \in \mathrm{Lock}(F, \mathcal{P}_0)$, and Theorem 2.5 yields a locking neighbourhood of (F, Ψ) in $C_0^r(M_0, M_0) \times \mathcal{P}_0$. In Lemma 5.5 we establish that this locking phenomenon descends to the original space: every F-invariant measure is supported on $M \cup \partial M_0$, and since the values of Ψ at ∂M_0 remain strictly below the maximum ergodic average, every (G, Ξ)-maximizing measure, for (G, Ξ) in the locking neighbourhood, is automatically supported on M, and hence coincides with the (g, ξ)-maximizing measure, where $g := G|_M$ and $\xi := \Xi|_M$. This extension procedure, and particularly Lemma 5.4, which provides norm bounds ensuring that the locking neighbourhood in $C_0^r(M_0, M_0) \times \mathcal{P}_0$ pulls back to one in $C^r(M, M) \times \mathcal{P}$, is a key novelty in this C^r one-dimensional case.

To prove Theorem E, we note that every logistic map satisfies $g_a(0) = g_a(1) = 0$, so $\mathcal{F} \subseteq C_0^1([0, 1], [0, 1])$, and the abstract framework of Proposition 5.1 applies directly. Density of hyperbolicity in $\mathcal{F}$ again follows from Graczyk–Świątek [GS97] and Lyubich [Ly97, p. 4], and Individual TPO for each hyperbolic $g_a \in \mathcal{F}$ follows from Lemma 5.2. The proof then proceeds by the same argument as for Theorem C: $\mathcal{F}$-stable hyperbolicity is inherited from Proposition 5.1 by restriction, and Theorem 2.5 yields the Joint TPO property.

Remark 1.3 (Relation to [HHJL25]**).** The present article is a companion to [HHJL25], which introduced the notion of Joint TPO and established it in three settings: open Lipschitz distance-expanding maps, Anosov diffeomorphisms, and beta-transformations. Here we clarify the relationship between the two works, and their somewhat different routes through related material.

Open distance-expanding maps. Let $\mathcal{E}(X)$ denote the space of open Lipschitz distance-expanding maps on a compact locally connected metric space X, as in [HHJL25]. Since $\Omega(T) = X$ for every $T \in \mathcal{E}(X)$, one can verify that conditions (RHE) and (ML) of Definition 2.1 hold for every $T \in \mathcal{E}(X)$: condition (RHE) holds with $K = 1$, using the distance-expanding property and the machinery of [HHJL25] establishing uniformity of expanding constants over an appropriate neighbourhood, while condition (ML) is precisely the Mañé lemma of [HHJL25], whose specific form is contained in [LS26]. Condition (IS), however, requires structural stability of $\mathcal{E}(X)$, and although in principle provable in this generality (as noted in [HHJL25]), it is not yet available in the literature for open distance-expanding maps on arbitrary compact locally connected metric spaces, indeed the method of [HHJL25] deliberately circumvents it, establishing Joint TPO instead via a more elementary Locally Connected Shadowing Lemma. Note that, once structural stability is established, the abstract framework of this article would yield only a *qualitative* Joint TPO statement, whereas [HHJL25] establishes a stronger *effective* Joint TPO theorem, with explicit quantitative bounds on the size of the joint locking neighbourhood in terms of the expanding constants and the Hölder seminorm of the potential function. The two approaches are thus complementary: the present article provides a broader axiomatic framework, while [HHJL25] achieves sharper quantitative conclusions via a more direct route.

Anosov diffeomorphisms. The approach of the present paper applies to C^r Axiom A diffeomorphisms with the no-cycle property (Theorem A), and in particular to Anosov diffeomorphisms (which are structurally stable, hence have the no-cycle condition). The paper [HHJL25] includes a self-contained proof of Joint TPO for Anosov diffeomorphisms as well, presented there as a deliberately less detailed preview of the present work.

Beta-transformations. The beta-transformation results of [HHJL25] lie entirely outside the scope of the present article, the two approaches having no overlap in this case. Beta-transformations $T_\beta\colon [0,1] \to [0,1]$, defined as $T_\beta(x) = \beta x \pmod 1$, are not continuous, so do not belong to any space of Lipschitz self-maps of a compact metric space in the sense required by Definition 2.1. Moreover, the sub-action for a beta-transformation in the Mañé lemma of [HHJL25] need not be continuous, so condition (ML) fails in the sense required here. The analysis in [HHJL25] of Joint TPO for beta-transformations is therefore independent of the methods presented here, and highlights that the Joint TPO phenomenon is not confined to the class of systems handled by the present article.

1.4. **Organisation of the article.** In Subsection 1.5 below, we collect some basic notation that is used throughout the article. In Section 2, we introduce an abstract framework accommodating a broad notion of hyperbolicity. Under these assumptions, we first establish a joint perturbation theorem (cf. Theorem 2.3) for Hölder potential functions ϕ, which generalises the results for Anosov diffeomorphisms of [HHJL25, Theorems 2.7 and 2.12]. We then use Theorem 2.3 to establish a C^1 analogue (cf. Theorem 2.4), which is used as an ingredient for establishing Joint TPO for $\mathcal{A}^r(M) \times C^1(M, \mathbb{R})$, and also give an important sufficient condition for a pair (f, ϕ) to belong to the interior of the joint locking set (cf. Theorem 2.5). In Section 3 we prove Theorem A, in Section 4 we prove Theorems B and C, and in Section 5 we prove Theorems D and E; each of these proofs, for a different family $\mathcal{F}$ of hyperbolic maps, proceeds by showing that the abstract $\mathcal{F}$-stable hyperbolicity of Section 2 is satisfied by the relevant family $\mathcal{F}$. Some of the more technical proofs are deferred to Appendix A.

1.5. **Notation.** Here we collect notation used throughout the article.

The set of positive integers is denoted by $\mathbb{N}$, and the set of nonnegative integers by $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$.

Let (X, d) be a metric space. For a subset $Y \subseteq X$, and $\varepsilon > 0$, the ε-neighbourhood of Y will be denoted by $B_X(Y, \varepsilon) \coloneqq \{x \in X : d(x, Y) < \varepsilon\}$, and if $Y = \{y\}$ is a singleton, we write $B_X(y, \varepsilon) \coloneqq B_X(\{y\}, \varepsilon)$.

In the definitions below, suppose moreover that X is compact.

For Lipschitz maps f, $g\colon X \to X$, we write

$$\mathrm{LIP}_d(f) \coloneqq \sup\{d(f(x), f(y))/d(x, y) : x, y \in X, x \neq y\}.$$

If there is no confusion regarding the metric, we shall omit the subscript d and simply write $\mathrm{LIP}(f)$. We denote the uniform distance between f and g by

$$d_\infty(f,g) \coloneqq \sup\{d(f(x),g(x)) : x \in X\}.$$

Given a map $f\colon X \to X$, for each $x \in X$ we denote the f-forward orbit of x by $\mathcal{O}^f(x) \coloneqq \{f^n(x)\}_{n=0}^{+\infty}$. If $\phi\colon X \to \mathbb{R}$ is a function, for every $n \in \mathbb{N}$, we denote

$$S_n^f \phi \coloneqq \sum_{i=0}^{n-1} \phi \circ f^i.$$

If $K \subseteq X$ is a finite subset, we denote the cardinality of K by $\operatorname{card} K$, and when K is nonempty we define the *gap* of K, denoted by $\Delta(K)$, to be $\Delta(K) \coloneqq +\infty$ if $\operatorname{card} K = 1$ and $\Delta(K) \coloneqq \min\{d(x,y) : x, y \in X, x \neq y\}$ if $\operatorname{card} K \geqslant 2$. The collection of all f-periodic orbits is denoted by $\mathrm{Per}(f)$. For every $x \in X$, we denote the Dirac measure at x by δ_x. If $\mathcal{O} \in \mathrm{Per}(f)$, we denote

$$\mu_{\mathcal{O}} \coloneqq \frac{1}{\operatorname{card} \mathcal{O}} \sum_{x \in \mathcal{O}} \delta_x.$$

For $\alpha \in (0,1]$, let $C^{0,\alpha}(X,\mathbb{R})$ denote the Banach space of real-valued α-Hölder functions defined on X, equipped with the norm

$$\|\phi\|_{\alpha,X} \coloneqq \|\phi\|_{\infty,X} + |\phi|_{\alpha,X},$$

where $\|\phi\|_{\infty,X} \coloneqq \sup_{x\in X}|\phi(x)|$ and $|\phi|_{\alpha,X} \coloneqq \sup\bigl\{\frac{|\phi(x)-\phi(y)|}{d(x,y)^\alpha} : x, y \in X, x \neq y\bigr\}$. Let $C^1(M,\mathbb{R})$ denote the Banach space of real-valued C^1 functions defined on a compact Riemannian manifold M, equipped with the norm

$$\|\phi\|_{C^1,M} \coloneqq \|\phi\|_{\infty,M} + \|\mathrm{D}\phi\|_{\infty,M},$$

where $\mathrm{D}\phi$ is the derivative of ϕ. We often omit the subscripts X or M if there is no risk of confusion.

2. Axiomatic hyperbolic framework and joint perturbation theorems

In this section, we first prove two *joint perturbation* theorems (Theorems 2.3 and 2.4) within an abstract hyperbolic framework, providing a more general formulation than [HHJL25, Theorems 2.7 and 2.12]. We then use these joint perturbation theorems to give a sufficient condition for a pair (f,ϕ) to lie in the interior of the joint locking set (see Theorem 2.5).

The abstract *stable hyperbolicity* condition underlying our theory is defined as follows.

Definition 2.1 ($\mathcal{F}$-stably hyperbolic maps). Let $\mathcal{F}$ be a topological space of Lipschitz self-maps on a compact metric space (X,d). We say that $f \in \mathcal{F}$ is *$\mathcal{F}$-stably hyperbolic* if the following conditions hold:

(**IS**) (**Intertwining stability.**) For every $\varepsilon > 0$, there exists a neighbourhood U of f in $\mathcal{F}$ such that if $g \in U$ then $\mathrm{LIP}(g) < 2\,\mathrm{LIP}(f)$, and there exist maps

$$h_g\colon \Omega(f) \to \Omega(g) \quad \text{and} \quad i_g\colon \Omega(g) \to \Omega(f)$$

satisfying $h_g \circ f = g \circ h_g$, $i_g \circ g = f \circ i_g$, $d_\infty(h_g,\mathrm{id}) < \varepsilon$, and $d_\infty(i_g,\mathrm{id}) < \varepsilon$.

(**RHE**) (**Robust hyperbolic estimates.**) There exists a neighbourhood U of f, and constants $K = K_U > 0$, $\delta = \delta_U > 0$, $\lambda = \lambda_U > 1$, such that for all $g \in U$, $x, y \in \Omega(g)$, and $n \in \mathbb{N}$, if

$$\max_{0\leqslant m\leqslant n} d(g^m(x),g^m(y)) < \delta,$$

then for all $0 \leqslant m \leqslant n$,

$$d(g^m(x),g^m(y)) \leqslant K\,\lambda^{-\min\{m,\,n-m\}}(d(x,y) + d(g^n(x),g^n(y))). \tag{2.1}$$

(**ML**) (**Mañé lemma.**) For every $\alpha \in (0,1]$, there exists a constant $L > 0$ such that for all $\phi \in C^{0,\alpha}(X,\mathbb{R})$ there exists a function $u \in C^{0,\alpha}(\Omega(f),\mathbb{R})$ satisfying

$$\phi + u - u \circ f \leqslant Q(f,\phi) \text{ on } \Omega(f), \quad \text{and} \quad |u|_\alpha \leqslant L\,|\phi|_\alpha.$$

Remark 2.2. Suppose that $\mathcal{F}$ is a topological space of Lipschitz self-maps on a compact metric space (X,d), that $f \in \mathcal{F}$ satisfies condition (RHE) in Definition 2.1, and that $\alpha \in (0,1]$. By direct calculation, we see that if x and y satisfy $\max_{0\leqslant m\leqslant n}\{d(g^m(x),g^m(y))\} < \rho \leqslant \delta$, then

$$\sum_{m=0}^{n} d(g^m(x),g^m(y))^\alpha \leqslant K^\alpha \sum_{m=0}^{n} (\lambda^{-m\alpha} + \lambda^{-(n-m)\alpha})(d(x,y) + d(g^n(x),g^n(y)))^\alpha \leqslant \frac{4K^\alpha \rho^\alpha \lambda^\alpha}{\lambda^\alpha - 1}.$$

We begin by establishing a joint perturbation theorem for Hölder potentials, providing a stronger and more general formulation than [HHJL25, Theorems 2.7 and 2.12].

Theorem 2.3 (Joint perturbation for Hölder potentials). *Let $\alpha \in (0,1]$, and let $\mathcal{F}$ be a topological space of Lipschitz self-maps on a compact metric space (X,d). If $f \in \mathcal{F}$ is $\mathcal{F}$-stably hyperbolic, and $\mathcal{O}$ is an f-periodic orbit, then there exist a neighbourhood U of f, and $C > 0$, such that for all $g \in U$, the following hold:*

(i) *The maps h_g and i_g in condition (IS) in Definition 2.1 exist.*

(ii) *If $\mathcal{O}_g := h_g(\mathcal{O})$, $d_g := \max\{d_\infty(h_g,\mathrm{id}),\, d_\infty(i_g,\mathrm{id})\}$, and a nonconstant function $\phi \in C^{0,\alpha}(X,\mathbb{R})$ satisfies $\mathcal{M}_{\max}(f,\phi) = \{\mu_\mathcal{O}\}$, then*

$$\{\mu_{\mathcal{O}_g}\} = \mathcal{M}_{\max}\big(g, \phi - 2C\,|\phi|_\alpha\, d_g^{\alpha/2}\, d(\cdot,\mathcal{O}_g)^\alpha + \xi\big)$$

for all $\xi \in C^{0,\alpha}(X,\mathbb{R})$ satisfying $|\xi|_\alpha \leqslant 5C|\phi|_\alpha d_g^{\alpha/2}$ and $\|\xi\|_\infty \leqslant |\phi|_\alpha d_g^\alpha$.

Proof. Fix an $\mathcal{F}$-stably hyperbolic map f and a periodic orbit $\mathcal{O}$ of f. By condition (IS) in Definition 2.1, there is a neighbourhood U_a of f such that for all $g \in U_a$, the maps h_g and i_g exist,

$$\mathrm{LIP}(g) \leqslant 2\,\mathrm{LIP}(f), \quad \text{and} \tag{2.2}$$

$$d_g < \min\{\Delta(\mathcal{O})/4,\ 1\}. \tag{2.3}$$

By condition (RHE) in Definition 2.1, there is a neighbourhood U_b of f and constants $K > 0$, $\delta > 0$, and $\lambda > 1$ such that (2.1) holds for $g \in U_b$. Let $L > 0$ be the constant obtained by condition (ML) in Definition 2.1. Then (i) holds in the neighbourhood

$$U := U_a \cap U_b. \tag{2.4}$$

Define constants

$$p_0 := \operatorname{card} \mathcal{O}, \tag{2.5}$$

$$r := \min\{\Delta(\mathcal{O})/(8\,\mathrm{LIP}(f)),\ \delta\}, \tag{2.6}$$

$$L_1 := 5 + 2L, \tag{2.7}$$

$$L_2 := 2(2K)^\alpha \lambda^\alpha/(\lambda^\alpha - 1), \tag{2.8}$$

$$L_3 := 1 + L + L(2\,\mathrm{LIP}(f))^\alpha, \quad \text{and} \tag{2.9}$$

$$C := \max\{1,\ 10L_2L_1r^{-\alpha}(2\,\mathrm{LIP}(f))^\alpha,\ 2(p_0 + L_2L_3)L_1r^{-\alpha}(2\,\mathrm{LIP}(f))^\alpha\}. \tag{2.10}$$

Fix a nonconstant $\phi \in C^{0,\alpha}(X,\mathbb{R})$ with $\mathcal{M}_{\max}(f,\phi) = \{\mu_\mathcal{O}\}$, $g \in U$, and $\xi \in C^{0,\alpha}(X,\mathbb{R})$, with

$$|\xi|_\alpha \leqslant 5C|\phi|_\alpha d_g^{\alpha/2} \quad \text{and} \quad \|\xi\|_\infty \leqslant |\phi|_\alpha d_g^\alpha. \tag{2.11}$$

In particular, since ϕ is nonconstant then $|\phi|_\alpha > 0$.

Since $h_g \circ f = g \circ h_g$, the image $\mathcal{O}_g := h_g(\mathcal{O})$ is a g-periodic orbit. For any two distinct $x, x' \in \mathcal{O}$, the triangle inequality and $d_\infty(h_g,\mathrm{id}) \leqslant d_g < \Delta(\mathcal{O})/4$ give

$$d(h_g(x),h_g(x')) \geqslant d(x,x') - 2d_g \geqslant \Delta(\mathcal{O}) - \Delta(\mathcal{O})/2 = \Delta(\mathcal{O})/2.$$

In particular h_g is injective on $\mathcal{O}$, so $p := \operatorname{card} \mathcal{O}_g = \operatorname{card} \mathcal{O} = p_0$ and $\Delta(\mathcal{O}_g) \geqslant \Delta(\mathcal{O})/2$. Hence C as defined in (2.10) satisfies

$$C \geqslant 2(p + L_2 L_3) L_1 r^{-\alpha} (2 \operatorname{LIP}(f))^{\alpha}. \tag{2.12}$$

If $\operatorname{LIP}(g)$ were strictly smaller than 1 then g would be a strict contraction, with $\Omega(g)$ consisting of a single fixed point q, and $\mu_{\mathcal{O}_g} = \delta_q$ the unique g-invariant probability measure; hence $\mathcal{M}_{\max}(g, \psi) = \{\delta_q\} = \{\mu_{\mathcal{O}_g}\}$ for every continuous ψ, and part (ii) would hold trivially. We therefore assume that $\operatorname{LIP}(g) \geqslant 1$ for the remainder of the proof.

From (2.6) and (2.2),

$$r \leqslant \Delta(\mathcal{O})/(8 \operatorname{LIP}(f)) \leqslant \Delta(\mathcal{O}_g)/(2 \operatorname{LIP}(g)). \tag{2.13}$$

By condition (ML) in Definition 2.1 applied to the map f, there exists $u \in C^{0,\alpha}(\Omega(f), \mathbb{R})$ satisfying

$$\psi := \overline{\phi} + u - u \circ f \leqslant 0 \text{ on } \Omega(f) \quad \text{ and } \quad |u|_\alpha \leqslant L |\phi|_\alpha. \tag{2.14}$$

where $\overline{\phi}$ denotes $\phi - Q(f, \phi)$. Define potentials

$$\Phi := \overline{\phi} - C\, |\phi|_\alpha\, d_g^{\alpha/2}\, d(\cdot, \mathcal{O}_g)^\alpha + \xi, \tag{2.15}$$

$$\psi_g := \overline{\phi} + u - u \circ g + \xi, \tag{2.16}$$

$$\Psi_g := \psi_g - C\, |\phi|_\alpha\, d_g^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha = \Phi + u - u \circ g. \tag{2.17}$$

Define constants

$$\tau := (3 + 2L) |\phi|_\alpha d_g^\alpha, \tag{2.18}$$

$$\eta := \int (\overline{\phi} + \xi)\, \mathrm{d}\mu_{\mathcal{O}_g} = \int \psi_g\, \mathrm{d}\mu_{\mathcal{O}_g} = \int \Psi_g\, \mathrm{d}\mu_{\mathcal{O}_g} = \frac{1}{p_0} \sum_{x \in \mathcal{O}} (\overline{\phi} + \xi)(h_g(x)), \tag{2.19}$$

where the second equality in (2.19) follows from (2.16) and the third from (2.17).

For every $x \in \Omega(g)$, we have $i_g(x) \in \Omega(f)$. So by (2.14), $\overline{\phi}(i_g(x)) \leqslant u(f(i_g(x))) - u(i_g(x))$. By the Hölder continuity of $\overline{\phi}$, and the fact that $d_\infty(i_g, \mathrm{id}) \leqslant d_g$,

$$\overline{\phi}(x) \leqslant \overline{\phi}(i_g(x)) + |\phi|_\alpha\, d(x, i_g(x))^\alpha \leqslant u(f(i_g(x))) - u(i_g(x)) + |\phi|_\alpha\, d_g^\alpha.$$

Using this, the fact that $i_g \circ g = f \circ i_g$ (see (i) and condition (IS) in Definition 2.1), and (2.16), (2.11), (2.14), (2.18), we obtain

$$\begin{aligned} \psi_g(x) &= \overline{\phi}(x) + u(x) - u(g(x)) + \xi(x) \leqslant u(x) - u(i_g(x)) + u(i_g(g(x))) - u(g(x)) + 2|\phi|_\alpha\, d_g^\alpha \\ &\leqslant |u|_\alpha\, d(x, i_g(x))^\alpha + |u|_\alpha\, d(i_g(g(x)), g(x))^\alpha + 2|\phi|_\alpha\, d_g^\alpha \leqslant (2L + 2)|\phi|_\alpha\, d_g^\alpha < \tau. \end{aligned} \tag{2.20}$$

By (2.19), (2.11), and (2.3), we have

$$\eta = \int \overline{\phi} + \xi\, \mathrm{d}\mu_{\mathcal{O}_g} \geqslant \frac{1}{p_0} \sum_{x \in \mathcal{O}} (\overline{\phi} \circ h_g)(x) - |\phi|_\alpha d_g^\alpha \geqslant \int \overline{\phi}\, \mathrm{d}\mu_{\mathcal{O}} - 2|\phi|_\alpha d_g^\alpha \geqslant -2|\phi|_\alpha d_g^\alpha. \tag{2.21}$$

Using (2.18), (2.21), and (2.7), we estimate

$$\tau - \eta \leqslant (5 + 2L)|\phi|_\alpha d_g^\alpha \leqslant L_1 |\phi|_\alpha d_g^\alpha. \tag{2.22}$$

By (2.19) and (2.20), we have $\eta = \int \psi_g\, \mathrm{d}\mu_{\mathcal{O}_g} < \tau$. So we can define

$$\rho := \left(C |\phi|_\alpha d_g^{\alpha/2} / (\tau - \eta) \right)^{-1/\alpha} > 0. \tag{2.23}$$

By (2.17), (2.20), and (2.23), we have

$$\Psi_g(x) < \tau - C\, |\phi|_\alpha\, d_g^{\alpha/2} \rho^\alpha = \eta \quad \text{ if } \quad x \notin B(\mathcal{O}_g, \rho). \tag{2.24}$$

Moreover, using (2.22) and (2.23), we estimate

$$\rho \leqslant (L_1/C)^{1/\alpha} \cdot d_g^{1/2}. \tag{2.25}$$

We wish to prove that $\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, \Phi)$. Since every g-invariant probability measure is supported on $\Omega(g)$ (cf. [Wa82, Theorem 6.15 (i) and Theorem 5.6 (i)]), this is equivalent to proving that $\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}\bigl(\Omega(g), g|_{\Omega(g)}, \Phi|_{\Omega(g)}\bigr)$. By (2.17), [Je19, Proposition 2.2], and (2.19), it suffices to establish that

$$\liminf_{n\to+\infty} \frac{1}{n} S_n^g \Psi_g(x) = \liminf_{n\to+\infty} \frac{1}{n} S_n^g \Phi(x) \leqslant \eta, \quad \text{for every } x \in \Omega(g). \tag{2.26}$$

Fix $x \in \Omega(g)$. We recursively construct a sequence $\{x_t\}_{t=1}^{s+1}$ of points in $\mathcal{O}^g(x)$ and a sequence $\{n_t\}_{t=1}^{s}$ of positive integers, for some $s \in \mathbb{N} \cup \{+\infty\}$, such that

$$x_{t+1} = g^{n_t}(x_t) \quad \text{and} \quad S_{n_t}^g \Psi_g(x_t) \leqslant n_t \eta \quad \text{for every } t \in \mathbb{N} \cap [0, s]. \tag{2.27}$$

Base step. Define $x_1 \coloneqq x$.

Recursive step. Assume that for some $t \in \mathbb{N}$, the finite sequences $\{x_i\}_{i=1}^{t}$ and $\{n_i\}_{i=1}^{t-1}$ are defined. Consider the following three cases.

Case A. Assume $x_t \notin B(\mathcal{O}_g, \rho)$. In this case we define $n_t \coloneqq 1$ and $x_{t+1} \coloneqq g(x_t)$. By (2.24),

$$S_{n_t}^g \Psi_g(x_t) = \Psi_g(x_t) < \eta = n_t \eta. \tag{2.28}$$

Case B. Assume $\mathcal{O}^g(x_t) \subseteq B(\mathcal{O}_g, r)$. Let $y \in \mathcal{O}_g$ be such that $d(x_t, y) = d(x_t, \mathcal{O}_g)$. By (2.13), we have $d(g(x_t), g(y)) \leqslant \mathrm{LIP}(g) d(x_t, y) \leqslant r \,\mathrm{LIP}(g) \leqslant 2^{-1}\Delta(\mathcal{O}_g)$, which implies $d(g(x_t), g(y)) = d(g(x_t), \mathcal{O}_g) \leqslant r$. By an inductive argument, we conclude that

$$d\bigl(g^l(x_t), g^l(y)\bigr) = d\bigl(g^l(x_t), \mathcal{O}_g\bigr) < r \quad \text{for all} \quad l \in \mathbb{N}. \tag{2.29}$$

By (2.17) and (2.29), we have

$$\liminf_{n\to+\infty} \frac{1}{n} S_n^g \Psi_g(x) = \liminf_{n\to+\infty} \frac{1}{n} S_n^g \Psi_g(x_t) \leqslant \liminf_{n\to+\infty} \frac{1}{n} \Biggl(S_n^g \Psi_g(y) + |\Psi_g|_\alpha \sum_{l=0}^{n-1} d\bigl(g^l(x_t), g^l(y)\bigr)^\alpha \Biggr).$$

As $r \leqslant \delta$ (see (2.6)), by Remark 2.2 we have

$$\sum_{l=0}^{n-1} d\bigl(g^l(x_t), g^l(y)\bigr)^\alpha \leqslant \frac{4K^\alpha r^\alpha \lambda^\alpha}{\lambda^\alpha - 1}.$$

Combining the above two inequalities and (2.19), we obtain $\liminf_{n\to+\infty} \frac{1}{n} S_n^g \Psi_g(x) \leqslant \eta$, which is precisely the required (2.26). This completes the recursive step.

Case C. Assume $\mathcal{O}^g(x_t) \nsubseteq B(\mathcal{O}_g, r)$ but $x_t \in B(\mathcal{O}_g, \rho)$. Let $y \in \mathcal{O}_g$ satisfy $d(x_t, y) = d(x_t, \mathcal{O}_g)$. Noting that $d_g \leqslant 1$ by (2.3), $C \geqslant 10L_2L_1r^{-\alpha}(2\,\mathrm{LIP}(f))^\alpha$ by (2.10), and our assumption that $\mathrm{LIP}(g) \geqslant 1$, using (2.25) and (2.2), we obtain that

$$\rho \leqslant (10L_2)^{-1/\alpha} \cdot 2^{-1} r / \mathrm{LIP}(f) \leqslant r / \mathrm{LIP}(g) \leqslant r. \tag{2.30}$$

Now define integers

$$N \coloneqq \min\bigl\{i \in \mathbb{Z} : i \geqslant -1,\, d\bigl(g^{i+1}(x_t), g^{i+1}(y)\bigr) \geqslant r\bigr\}, \tag{2.31}$$

$$m \coloneqq \max\bigl\{i \in \mathbb{Z} : 1 \leqslant i \leqslant N,\, d\bigl(g^{i-1}(x_t), g^{i-1}(y)\bigr) < \rho\bigr\}, \tag{2.32}$$

where the existence of N follows from the assumptions of this case, and (2.30) implies that $1 \leqslant N$, so m is well defined. Using (2.31), and an argument analogous to the one used to prove (2.29) in Case B, yields

$$d\bigl(g^i(x_t), g^i(y)\bigr) = d\bigl(g^i(x_t), \mathcal{O}_g\bigr) < r \quad \text{for all } 0 \leqslant i \leqslant N. \tag{2.33}$$

By definition of N (cf. (2.31)), we have $d\bigl(g^N(x_t), g^N(y)\bigr) < r$ but $d\bigl(g^{N+1}(x_t), g^{N+1}(y)\bigr) \geqslant r$. Since $d\bigl(g^{N+1}(x_t), g^{N+1}(y)\bigr) \leqslant \mathrm{LIP}(g)\, d\bigl(g^N(x_t), g^N(y)\bigr)$, we obtain

$$d\bigl(g^N(x_t), g^N(y)\bigr) \geqslant r / \mathrm{LIP}(g) \geqslant r / (2\,\mathrm{LIP}(f)) \coloneqq r_0, \tag{2.34}$$

where the last inequality uses (2.2).

In this case we define $n_t \coloneqq N+1$ and $x_{t+1} \coloneqq g^{N+1}(x_t)$.

Next, we estimate $S^g_{n_t}\Psi_g(x_t)$ so as to deduce (2.27). Since $\Psi_g \leqslant \psi_g$, direct calculation gives

$$\begin{aligned} S^g_{n_t}\Psi_g(x_t) &\leqslant S^g_m\psi_g(x_t) + S^g_{N-m}\Psi_g(g^m(x_t)) + \Psi_g\big(g^N(x_t)\big) \\ &\leqslant S^g_m\psi_g(y) + |S^g_m\psi_g(x_t) - S^g_m\psi_g(y)| + S^g_{N-m}\Psi_g(g^m(x_t)) + \Psi_g\big(g^N(x_t)\big). \end{aligned} \tag{2.35}$$

Next we estimate the four terms on the righthand side of (2.35) separately. For the first term, write $m = pq + l$ for some $q \in \mathbb{N}_0$ and $0 \leqslant l \leqslant p-1$, then by (2.19) and (2.20), we have

$$S^g_m\psi_g(y) = qS^g_p\psi_g(y) + S^g_l\psi_g(y) \leqslant pq\eta + l\tau \leqslant m\eta + (p-1)(\tau-\eta). \tag{2.36}$$

For the second term, since $d(g^i(x_t), g^i(y)) < r \leqslant \delta$ for all $0 \leqslant i \leqslant m-1$ (by (2.33)), and the endpoints of this orbit segment satisfy $d(x_t, y) < \rho$ and $d(g^{m-1}(x_t), g^{m-1}(y)) < \rho$ (by the definitions of x_t and m), applying condition (RHE) in Definition 2.1 to this segment yields

$$d(g^i(x_t), g^i(y)) < 2K\lambda^{-\min\{i,\, m-1-i\}}\rho \qquad \text{for each } 0 \leqslant i \leqslant m-1.$$

Therefore, using (2.16), condition (ML) in Definition 2.1, (2.2), (2.11), (2.8), and (2.9),

$$\begin{aligned} |S^g_m\psi_g(x_t) - S^g_m\psi_g(y)| &\leqslant |\psi_g|_\alpha \sum_{i=0}^{m-1} d\big(g^i(x_t), g^i(y)\big)^\alpha \\ &\leqslant |\psi_g|_\alpha (2K\rho)^\alpha \frac{2\lambda^\alpha}{\lambda^\alpha - 1} \leqslant L_2\rho^\alpha\,|\phi|_\alpha\big(L_3 + 5Cd_g^{\alpha/2}\big). \end{aligned} \tag{2.37}$$

For the third term, by (2.32), (2.33), and (2.24), we have

$$S^g_{N-m}\Psi_g(g^m(x_t)) \leqslant (N-m)\eta. \tag{2.38}$$

For the fourth term, by (2.17), (2.20), (2.31), (2.33), and (2.34), we have

$$\Psi_g\big(g^N(x_t)\big) \leqslant \tau - C|\phi|_\alpha d_g^{\alpha/2} d\big(g^N(x_t), \mathcal{O}_g\big)^\alpha \leqslant \tau - C|\phi|_\alpha d_g^{\alpha/2} r_0^\alpha. \tag{2.39}$$

Finally, combining (2.35)–(2.39), and using (2.22) and (2.25), we obtain

$$\begin{aligned} S^g_{n_t}\Psi_g(x_t) - n_t\eta &\leqslant p(\tau-\eta) + \big(L_3 + 5Cd_g^{\alpha/2}\big)L_2\rho^\alpha|\phi|_\alpha - C|\phi|_\alpha d_g^{\alpha/2} r_0^\alpha \\ &\leqslant pL_1|\phi|_\alpha d_g^\alpha + L_1L_2L_3|\phi|_\alpha d_g^{\alpha/2} + 5CL_2\rho^\alpha d_g^{\alpha/2}|\phi|_\alpha - Cr_0^\alpha|\phi|_\alpha d_g^{\alpha/2}. \end{aligned} \tag{2.40}$$

Since $d_g \leqslant 1$ (cf. (2.3)) and $C \geqslant 2(p + L_2L_3)L_1 r_0^{-\alpha}$ (cf. (2.12)), we obtain

$$pL_1|\phi|_\alpha d_g^\alpha + L_1L_2L_3|\phi|_\alpha d_g^{\alpha/2} \leqslant 2^{-1}Cr_0^\alpha|\phi|_\alpha d_g^{\alpha/2}. \tag{2.41}$$

Using (2.30) and (2.34), we obtain $5CL_2\rho^\alpha d_g^{\alpha/2}|\phi|_\alpha \leqslant 2^{-1}Cr_0^\alpha|\phi|_\alpha d_g^{\alpha/2}$. This, combined with (2.40) and (2.41), gives

$$S^g_{n_t}\Psi_g(x_t) \leqslant n_t\eta. \tag{2.42}$$

So the required inequality (2.27) holds, and therefore the recursive step is complete.

If the recursion never terminates (i.e., $s = +\infty$, meaning Case B never occurs), then setting $N_t \coloneqq \sum_{i=1}^t n_i$ for every $t \in \mathbb{N}$, by (2.28) and (2.42), we have

$$\liminf_{n\to+\infty} \frac{1}{n} S^g_n\Psi_g(x) \leqslant \liminf_{t\to+\infty} \frac{1}{N_t}\sum_{i=1}^t S^g_{n_i}\Psi_g(x_i) \leqslant \lim_{t\to+\infty}\frac{1}{N_t}\sum_{i=1}^t n_i\eta \coloneqq \eta.$$

Thus (2.26) holds, so $\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, \Phi)$. Now define $h \coloneqq -C\,|\phi|_\alpha\, d_g^{\alpha/2}\, d(\cdot, \mathcal{O}_g)^\alpha$. Since $h \leqslant 0$ with $h = 0$ exactly on $\mathcal{O}_g$, the maximum ergodic average $Q(g, h) = 0$ is achieved precisely by those g-invariant measures supported on $\mathcal{O}_g$. Since $\mathcal{O}_g$ is a g-periodic orbit, the unique g-invariant measure supported on $\mathcal{O}_g$ is $\mu_{\mathcal{O}_g}$, so $\mathcal{M}_{\max}(g, h) = \{\mu_{\mathcal{O}_g}\}$. This, combined with the fact that

$\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, \Phi)$, gives $\mathcal{M}_{\max}(g, \Phi+h) = \{\mu_{\mathcal{O}_g}\}$. As $\Phi+h = \phi - Q(f,\phi) - 2C\,|\phi|_\alpha\, d_g^{\alpha/2} d(\cdot, \mathcal{O}_g)^\alpha + \xi$, we conclude that $\{\mu_{\mathcal{O}_g}\} = \mathcal{M}_{\max}\bigl(g, \phi - 2C\,|\phi|_\alpha\, d_g^{\alpha/2}\, d(\cdot,\mathcal{O}_g)^\alpha + \xi\bigr)$, as required. □

Now we prove a C^1 analogue of Theorem 2.3, i.e., the joint perturbation theorem for C^1 potentials on a compact smooth manifold M.

Theorem 2.4 (Joint perturbation for C^1 potentials). *Let M be a compact smooth manifold equipped with a Riemannian metric, and let $\mathcal{F}$ be a topological space consisting of Lipschitz self-maps of M. If $f \in \mathcal{F}$ is $\mathcal{F}$-stably hyperbolic, and $\mathcal{O}$ is an f-periodic orbit, then there exist a neighbourhood U of f, and $C > 0$, such that for all $g \in U$, the following hold:*

(i) *The maps h_g and i_g in condition (IS) in Definition 2.1 exist.*

(ii) *Denote $\mathcal{O}_g := h_g(\mathcal{O})$ and $d_g := \max\{d_\infty(h_g, \mathrm{id}),\, d_\infty(i_g, \mathrm{id})\}$. There exists $v_g \in C^1(M,\mathbb{R})$ such that*
$$\|\mathrm{D}v_g\|_\infty \leqslant 3Cd_g^{1/2} \quad \text{and} \quad \mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, v_g), \tag{2.43}$$
and for all $\phi \in C^1(M,\mathbb{R})$ with $\mathcal{M}_{\max}(f,\phi) = \{\mu_{\mathcal{O}}\}$,
$$\mathcal{M}_{\max}(g, \phi + \|\mathrm{D}\phi\|_\infty v_g) = \{\mu_{\mathcal{O}_g}\}.$$

Proof. Without loss of generality, assume that $\operatorname{diam} M = 1$. Let the neighbourhood U and constant $C > 0$ be as given by Theorem 2.3 for the case $\alpha := 1$. Then (i) follows from Theorem 2.3 (i).

Fix $g \in U$ and $\phi \in C^1(M,\mathbb{R})$ with $\mathcal{M}_{\max}(f,\phi) = \{\mu_{\mathcal{O}}\}$. By [HLMXZ25, Theorem 2.7], there exists $w \in C^1(M,\mathbb{R})$ satisfying
$$\|\mathrm{D}w\|_\infty < 3/2 \quad \text{and} \quad \|w + d(\cdot,\mathcal{O}_g)\|_\infty < d_g^{1/2}/(2C). \tag{2.44}$$
Define $\xi := 2C\|\mathrm{D}\phi\|_\infty d_g^{1/2}(d(\cdot,\mathcal{O}_g) + w)$. By (2.44), we obtain
$$|\xi|_1 \leqslant 5C\|\mathrm{D}\phi\|_\infty d_g^{1/2} \quad \text{and} \quad \|\xi\|_\infty \leqslant \|\mathrm{D}\phi\|_\infty d_g. \tag{2.45}$$
Hence denoting $v_g := 2Cd_g^{1/2} w$, by Theorem 2.3 (ii) and (2.44), we obtain $\mathcal{M}_{\max}(g, \phi + \|\mathrm{D}\phi\|_\infty v_g) = \{\mu_{\mathcal{O}_g}\}$ and $\|\mathrm{D}v_g\|_\infty \leqslant 3Cd_g^{1/2}$.

So it suffices to show that $\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, v_g)$.

For each $n \in \mathbb{N}$, define $\phi_n := \max\{-1/n,\, -d(\cdot,\mathcal{O})\} \in C^{0,1}(M,\mathbb{R})$. Then clearly for each $n \in \mathbb{N}$,
$$|\phi_n|_1 = 1, \qquad -1/n \leqslant \phi_n \leqslant 0, \qquad \text{and } \mathcal{O} = \phi_n^{-1}(0). \tag{2.46}$$
Obviously, $\mathcal{M}_{\max}(f,\phi_n) = \{\mu_{\mathcal{O}}\}$ for each $n \in \mathbb{N}$. Fix arbitrary $\mu \in \mathcal{M}(M,g)$ and $n \in \mathbb{N}$. By Theorem 2.3 (ii) and (2.44), we have $\mathcal{M}_{\max}(g, \phi_n + v_g) = \{\mu_{\mathcal{O}_g}\}$. So $\int(\phi_n + v_g)\,\mathrm{d}\mu_{\mathcal{O}_g} > \int(\phi_n + v_g)\,\mathrm{d}\mu$. By (2.46), $\int \phi_n\,\mathrm{d}\mu \geqslant -\frac{1}{n} \geqslant \int\phi_n\,\mathrm{d}\mu_{\mathcal{O}_g} - \frac{1}{n}$. So we have $\int v_g\,\mathrm{d}\mu_{\mathcal{O}_g} > \int v_g\,\mathrm{d}\mu - \frac{1}{n}$ for all $n \in \mathbb{N}$, and hence $\int v_g\,\mathrm{d}\mu_{\mathcal{O}_g} \geqslant \int v_g\,\mathrm{d}\mu$. This means that $\mu_{\mathcal{O}_g} \in \mathcal{M}_{\max}(g, v_g)$, as required. □

To conclude this section, we now deduce a sufficient condition for a pair (f,ϕ) to belong to the interior of the joint locking set.

Theorem 2.5 (Interior Condition). *Let $\alpha \in (0,1]$, (X,d) be a compact metric space, and (M,ρ) be a compact smooth manifold equipped with a Riemannian metric.*

(i) *If $\mathcal{F}$ is a topological space consisting of Lipschitz self-maps of X, and $f \in \mathcal{F}$ is $\mathcal{F}$-stably hyperbolic, and $(f,\phi) \in \operatorname{Lock}\bigl(\mathcal{F}, C^{0,\alpha}(X,\mathbb{R})\bigr)$ where ϕ is nonconstant, then (f,ϕ) belongs to the interior of the joint locking set $\operatorname{Lock}\bigl(\mathcal{F}, C^{0,\alpha}(X,\mathbb{R})\bigr)$.*

(ii) *If $\mathcal{F}$ is a topological space consisting of Lipschitz self-maps of M, and $f \in \mathcal{F}$ is $\mathcal{F}$-stably hyperbolic, and $(f,\phi) \in \operatorname{Lock}\bigl(\mathcal{F}, C^1(M,\mathbb{R})\bigr)$ where ϕ is nonconstant, then (f,ϕ) belongs to the interior of the joint locking set $\operatorname{Lock}\bigl(\mathcal{F}, C^1(M,\mathbb{R})\bigr)$.*

Proof. Since the proofs of (i) and (ii) share a common core, here we prove the more complicated (ii) in detail and give a more abbreviated proof of (i).

(ii) Fix $(f,\phi)\in \operatorname{Lock}\big(\mathcal{F},C^1(M,\mathbb{R})\big)$ such that f is $\mathcal{F}$-stably hyperbolic. Since ϕ is not a constant function, $\|\mathrm{D}\phi\|_\infty>0$. Let us write $B(\phi,r)\coloneqq\big\{\psi\in C^1(M,\mathbb{R}):\|\psi-\phi\|_{C^1}<r\big\}$.

Let $\mathcal{O}$ be the f-periodic orbit satisfying $\mathcal{M}_{\max}(f,\phi)=\{\mu_{\mathcal{O}}\}$. Then there exists $\theta>0$ such that if $\psi\in B(\phi,\theta)$ then $\mathcal{M}_{\max}(f,\psi)=\{\mu_{\mathcal{O}}\}$. We can assume without loss of generality that θ is small enough (i.e., $\theta\leqslant 2^{-1}\|\mathrm{D}\phi\|_\infty$) such that

$$\frac{1}{2}\|\mathrm{D}\phi\|_\infty\leqslant\|\mathrm{D}\psi\|_\infty\leqslant\frac{3}{2}\|\mathrm{D}\phi\|_\infty\qquad\text{for all }\psi\in B(\phi,\theta).\tag{2.47}$$

Applying Theorem 2.3 (to f and $\mathcal{O}$), let the neighbourhood U of f, and the constant $C>0$, be as in that theorem. By condition (IS) in Definition 2.1, there exists a neighbourhood $V\subseteq U$ of f such that

$$d_g<(9C(1+\operatorname{diam}(M))\|\mathrm{D}\phi\|_\infty/\theta)^{-2}\qquad\text{for all }g\in V.\tag{2.48}$$

For every $g\in V$, let v_g be the function obtained from Theorem 2.4 (ii), and assume without loss of generality that $v_g(x)=0$ for some $x\in M$. By Theorem 2.4 (ii) and (2.48), we have

$$\begin{aligned}\|v_g\|_{C^1}=\|v_g\|_\infty+\|\mathrm{D}v_g\|_\infty&\leqslant(1+\operatorname{diam}(M))\|\mathrm{D}v_g\|_\infty\\&\leqslant 3Cd_g^{1/2}(1+\operatorname{diam}(M))<\theta/(3\|\mathrm{D}\phi\|_\infty).\end{aligned}\tag{2.49}$$

By Theorem 2.4, for all $g\in V$ and $\psi\in B(\phi,\theta)$, we have

$$\mathcal{M}_{\max}(g,\psi+\|\mathrm{D}\psi\|_\infty v_g)=\{\mu_{\mathcal{O}_g}\}\subseteq\mathcal{M}_{\max}(g,v_g).\tag{2.50}$$

Now fix $(g,\psi)\in V\times B(\phi,\theta/2)$, and set

$$\psi'\coloneqq\psi-(3/2)\|\mathrm{D}\phi\|_\infty v_g.\tag{2.51}$$

By (2.51) and (2.49), we obtain $\|\psi-\psi'\|_{C^1}\leqslant(3/2)\|\mathrm{D}\phi\|_\infty\|v_g\|_{C^1}\leqslant\theta/2$. Using this, and the fact that $\psi\in B(\phi,\theta/2)$, we deduce that $\psi'\in B(\phi,\theta)$. Thus applying (2.47) to ψ and ψ', we obtain

$$\|\mathrm{D}\psi\|_\infty\geqslant\|\mathrm{D}\phi\|_\infty/2\geqslant\|\mathrm{D}\psi'\|_\infty/3.\tag{2.52}$$

From (2.50) we obtain that $\{\mu_{\mathcal{O}_g}\}=\mathcal{M}_{\max}(g,\psi'+\|\mathrm{D}\psi'\|_\infty v_g)$. By (2.51),

$$\psi=\psi'+(3/2)\|\mathrm{D}\phi\|_\infty v_g=\psi'+\|\mathrm{D}\psi'\|_\infty v_g+((3/2)\|\mathrm{D}\phi\|_\infty-\|\mathrm{D}\psi'\|_\infty)v_g.$$

By (2.52), $\mu_{\mathcal{O}_g}\in\mathcal{M}_{\max}(g,((3/2)\|\mathrm{D}\phi\|_\infty-\|\mathrm{D}\psi'\|_\infty)v_g)$. Hence $\mathcal{M}_{\max}(g,\psi)=\{\mu_{\mathcal{O}_g}\}$.

Therefore, we have established that the neighbourhood $V\times B(\phi,\theta/2)$ of (f,ϕ) is contained in $\operatorname{Lock}(\mathcal{F},C^1(M,\mathbb{R}))$, so (ii) is proved.

(i) Setting $v_g\coloneqq-2C|\phi|_\alpha d_g^{\alpha/2}d(\cdot,\mathcal{O}_g)^\alpha$, and using Theorem 2.3, we obtain properties of v_g analogous to Theorem 2.4 (ii), then (i) follows by using the same argument as in the proof of (ii). □

Remark 2.6. (i) The role of the nonwandering set $\Omega(f)$ in Definition 2.1 can be generalised as follows. Let $\varsigma\colon\mathcal{F}\to 2^X$ be a map such that for every $f\in\mathcal{F}$, $f(\varsigma(f))\subseteq\varsigma(f)$, and every f-invariant measure is supported on $\varsigma(f)$. If $\Omega(f)$ is replaced by $\varsigma(f)$ throughout Definition 2.1, then Theorem 2.5 remains valid (a fact we shall use in the proof of Theorem D). This is because the only property of $\Omega(f)$ used in the proofs of Theorems 2.3, 2.4, and 2.5 was that every f-invariant measure on M is an invariant measure for the subsystem $(\Omega(f),f)$.

(ii) Using [Je19, Proposition 2.2], we can see that if $f\in\mathcal{F}$ satisfies conditions (IS) and (ML) in Definition 2.1 (or their modifications from (i) above), and $\phi\in\operatorname{Lock}\big(\mathcal{F},C^{0,\alpha}(X,\mathbb{R})\big)$ for some $\alpha\in(0,1]$, then $\lim_{g\to f}Q(g,\phi)=Q(f,\phi)$. Indeed from (2.20) we see that $\limsup_{g\to f}Q(g,\phi)\leqslant Q(f,\phi)$, while from Theorem 2.3 (ii) we see that

$$\liminf_{g\to f}Q(g,\phi)\geqslant\liminf_{g\to f}Q\big(g,\phi-2Cd_g^{\alpha/2}d(\cdot,\mathcal{O}_g)^\alpha\big)=\liminf_{g\to f}\int\phi\,\mathrm{d}\mu_{\mathcal{O}_g},$$

and $\lim_{g\to f} d(h_g, \mathrm{id}|_{\Omega(f)}) = 0$, so $\lim_{g\to f} \int \phi \,\mathrm{d}\mu_{\mathcal{O}_g} = \int \phi \,\mathrm{d}\mu_{\mathcal{O}}$, and thus $\liminf_{g\to f} Q(g,\phi) \geqslant Q(f,\phi)$.

3. Axiom A diffeomorphisms: proof of Theorem A

If (X,d) is a compact metric space, and $f\colon X \to X$ a continuous map, recall that a point $x \in X$ is said to be *wandering* if there is a neighbourhood U of x such that $f^n(U)\cap U = \emptyset$ for each $n \in \mathbb{N}$, and the *nonwandering set* $\Omega(f)$ of f is defined as

$$\Omega(f) \coloneqq X \smallsetminus \{x \in X : x \text{ is wandering}\}.$$

It is well known that the nonwandering set is nonempty and compact, and every f-invariant probability measure has its support contained in $\Omega(f)$. Moreover, if f is a homeomorphism then $\Omega(f)$ is f-invariant, in the sense that $f(\Omega(f)) = \Omega(f)$ (cf. [URM22, Theorem 1.4.9 (e)]). A compact f-invariant subset $\Lambda \subseteq X$ is called *(topologically) transitive* if $\mathcal{O}^f(x)$ is dense in Λ for some $x \in \Lambda$.

Now let M be a compact smooth manifold (without boundary), with Riemannian metric $|\cdot|$ on M, and the induced distance function d.

Definition 3.1 (Axiom A diffeomorphisms). Let $f\colon M \to M$ be a diffeomorphism, and $\mathrm{D}f$ its derivative. An f-invariant set Λ is called *hyperbolic* if for each $x \in \Lambda$, the tangent space T_xM can be split into a direct sum $T_xM = E^s(x) \oplus E^u(x)$, where the subspaces $E^s(x)$ and $E^u(x)$ are $\mathrm{D}f$-invariant, i.e., $\mathrm{D}f(x)E^s(x) = E^s(f(x))$ and $\mathrm{D}f(x)E^u(x) = E^u(f(x))$, and there exist constants $C \geqslant 1$ and $0 < \xi < 1$ such that

$$|\mathrm{D}f^n(x)(u)| \leqslant C\xi^n|u| \quad \text{for all } x \in \Lambda,\ u \in E^s(x), n \in \mathbb{N}_0,$$
$$|\mathrm{D}f^{-n}(x)(u)| \leqslant C\xi^n|u| \quad \text{for all } x \in \Lambda,\ u \in E^u(x), n \in \mathbb{N}_0.$$

If the nonwandering set $\Omega(f)$ is hyperbolic, and the set of f-periodic points is dense in $\Omega(f)$, then f is called an *Axiom A diffeomorphism*.

Let $f\colon M \to M$ be an Axiom A diffeomorphism. The spectral decomposition theorem (see [Sm67] and cf. e.g. [Wen16, Theorem 5.2]) states that $\Omega(f)$ is uniquely decomposed into finitely many disjoint compact transitive subsets, i.e., $\Omega(f) = \bigcup_{i=1}^{k} \Omega_i$, where Ω_i is compact and transitive for each $1 \leqslant i \leqslant k$, and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. For each $1 \leqslant i \leqslant k$, the *stable manifold* $W^s(\Omega_i)$ and *unstable manifold* $W^u(\Omega_i)$ are defined as

$$W^s(\Omega_i) \coloneqq \bigl\{x \in X : \lim_{n\to+\infty} d(f^n(x), \Omega_i) = 0\bigr\},$$
$$W^u(\Omega_i) \coloneqq \bigl\{x \in X : \lim_{n\to+\infty} d(f^{-n}(x), \Omega_i) = 0\bigr\}.$$

For $1 \leqslant i,\, j \leqslant k$, we write $\Omega_i \rightharpoonup \Omega_j$ if

$$W^u(\Omega_i) \cap W^s(\Omega_j) \nsubseteq \Omega(f).$$

The diffeomorphism f is said to have a *cycle* if there exist $1 \leqslant i_1,\, \dots,\, i_m \leqslant k$ such that

$$\Omega_{i_1} \rightharpoonup \Omega_{i_2} \rightharpoonup \cdots \rightharpoonup \Omega_{i_m} \rightharpoonup \Omega_{i_1}.$$

We say that f has the *no-cycle property* if it does not have any cycles.

Fix $r \in \mathbb{N}$ and $\alpha \in (0,1]$. Suppose that M is a compact smooth manifold equipped with a Riemannian metric, and recall that $\mathcal{A}^r(M)$ denotes the space of those C^r Axiom A diffeomorphisms on M with the no-cycle property. We first prove that each $f \in \mathcal{A}^r(M)$ is $\mathcal{A}^r(M)$-stably hyperbolic, by means of the following three results (Lemmas 3.2, 3.3, and 3.5).

Firstly, it is well known that an Axiom A diffeomorphism with the no-cycle condition satisfies the following enhanced version of condition (IS) in Definition 2.1:

Lemma 3.2. *Suppose M is a compact smooth manifold equipped with a Riemannian metric, $r \in \mathbb{N}$, and $f \in \mathcal{A}^r(M)$. Then f satisfies condition (IS) in Definition 2.1 for $\mathcal{F} \coloneqq \mathcal{A}^r(M)$. Moreover, if $g \in \mathcal{A}^r(M)$ is sufficiently close to f then h_g can be taken to be a homeomorphism.*

Proof. By [Wen16, Theorem 5.8], for every $r \in \mathbb{N}$, an Axiom A diffeomorphism $f \in \mathcal{A}^r(M)$ satisfying the no-cycle condition has C^r *ε-Ω-stability*, i.e., for all $\varepsilon > 0$, there exists a C^r neighbourhood $U \subseteq \mathrm{Diff}^r(M)$ such that for every $g \in U$, the restriction $g|_{\Omega(g)}$ is topologically conjugate to $f|_{\Omega(f)}$, and the conjugacy is ε-close to the identity. ☐

Regarding condition (RHE) in Definition 2.1, we have:

Lemma 3.3. *Let M be a compact smooth manifold equipped with a Riemannian metric $|\cdot|$, and the induced distance function d. If $r \in \mathbb{N}$ and $f \in \mathcal{A}^r(M)$, then f satisfies condition (RHE) in Definition 2.1 for $\mathcal{F} := \mathcal{A}^r(M)$.*

Proof. See Appendix A. ☐

Remark 3.4. The single-map version of Lemma 3.3, in which the constants K, δ, λ depend on f alone, is the content of [KH95, Proposition 6.4.16]. The essential new content of Lemma 3.3 is that the constants K, δ, λ of condition (RHE) in Definition 2.1 can be chosen *uniformly* over a neighbourhood of f in $\mathcal{A}^r(M)$.

Regarding condition (ML) in Definition 2.1, we have:

Lemma 3.5. *Suppose M is a compact smooth manifold equipped with a Riemannian metric. Every map in $\mathcal{A}^1(M)$ satisfies condition (ML) in Definition 2.1.*

Proof. Fix $f \in \mathcal{A}^1(M)$ and $\alpha \in (0,1]$. By [Wen16, Theorem 5.3], the nonwandering set $\Omega(f)$ is *locally maximal*, in the sense that there exists a compact neighbourhood K of $\Omega(f)$ such that $\Omega(f) = \bigcap_{n\in\mathbb{Z}} f^n(K)$. In the Lipschitz case $\alpha = 1$, the lemma follows immediately from [STY24, Theorem 1.3], while if $\alpha \in (0,1)$ we can proceed by adapting arguments in [STY24]. More specifically, using the same arguments as in [STY24, Section 2] (specifically, Step 5 in the proof there), we obtain a modified form of [STY24, Corollary 1.6] with [STY24, (1.5)] replaced by the inequality

$$\sum_{i=1}^{n} d\big(x_i, f^i(p)\big)^\alpha \leqslant C \sum_{k=1}^{n} d(f(x_{k-1}), x_k)^\alpha,$$

where C is a constant depending only on f and M. By then using a modified discrete Lax–Oleinik operator (cf. [STY24, Definition 3.1]), with the expression (3.1) of [STY24] replaced by

$$T[u](x) := \inf_{x'\in K} \{u(x') + \phi(x') - \overline{\phi}_\Lambda + C d(f(x'), x)^\alpha\} \qquad \text{for } x \in K,$$

the result then follows using arguments analogous to those in [STY24, Sections 3 and 4]. ☐

As a corollary of [HLMXZ25, Theorems 1.1 and 1.2], if $f\colon M \to M$ is an Axiom A diffeomorphism, then $(f, \mathcal{P})$ satisfies the TPO property. More precisely, we have:

Proposition 3.6 (Individual TPO for Axiom A diffeomorphisms). *Suppose M is a compact smooth manifold equipped with a Riemannian metric, $\alpha \in (0,1]$, and $\mathcal{P}$ is either $C^{0,\alpha}(M,\mathbb{R})$ or $C^1(M,\mathbb{R})$. For every Axiom A diffeomorphism f on M, the locking set* $\mathrm{Lock}(f,\mathcal{P})$ *is an open dense subset of $\mathcal{P}$.*

Proof. Let $f\colon M \to M$ be an Axiom A diffeomorphism. Clearly, $\mathrm{Lock}(f,\mathcal{P})$ is an open subset of $\mathcal{P}$, so it suffices to prove that $\mathrm{Lock}(f,\mathcal{P})$ is dense in $\mathcal{P}$.

We consider the dynamical system $(\Omega(f), f|_{\Omega(f)})$. By [KH95, Propositions 6.4.15 & 6.4.16], and Lemma 3.5, $(\Omega(f), f|_{\Omega(f)})$ satisfies the (ACP), (EI), and (NLP) properties of [HLMXZ25]. Note that for every $\phi \in \mathcal{P}$, $\mathcal{M}_{\max}(M, f, \phi) = \mathcal{M}_{\max}(\Omega(f), f|_{\Omega(f)}, \phi|_{\Omega(f)})$.

If $\mathcal{P} = C^1(M,\mathbb{R})$, then by [HLMXZ25, Theorem 1.2], $\mathrm{Lock}\big(f, C^1(M,\mathbb{R})\big)$ is dense in $C^1(M,\mathbb{R})$.

Assume that $\mathcal{P} = C^{0,\alpha}(M,\mathbb{R})$. Fix $\varepsilon > 0$ and $\phi \in C^{0,\alpha}(M,\mathbb{R})$. By [HLMXZ25, Theorem 1.1], there exists $\psi \in C^{0,\alpha}(\Omega(f))$ with $\|\psi\|_{\alpha,\Omega(f)} < \varepsilon$, $\delta > 0$, and a periodic orbit $\mathcal{O}$ of f such that if $\xi \in C^{0,\alpha}(\Omega(f))$ with $\|\xi\|_{\alpha,\Omega(f)} < \delta$ then $\mathcal{M}_{\max}(\Omega(f), f|_{\Omega(f)}, \phi|_{\Omega(f)} + \psi + \xi) = \{\mu_{\mathcal{O}}\}$. By [Wea18, Theorem 1.33], there exists $\widetilde{\psi} \in C^{0,\alpha}(M,\mathbb{R})$ with $\|\widetilde{\psi}\|_{\alpha,M} \leqslant \varepsilon$ and $\widetilde{\psi}|_{\Omega(f)} = \psi$. For every $\zeta \in C^{0,\alpha}(M,\mathbb{R})$ with $\|\zeta\|_{\alpha,M} < \delta$, then $\|\zeta|_{\Omega(f)}\|_{\alpha,\Omega(f)} < \delta$. So we have

$$\mathcal{M}_{\max}\big(M, f, \phi + \widetilde{\psi} + \zeta\big) = \mathcal{M}_{\max}\big(\Omega(f), f|_{\Omega(f)}, (\phi + \psi + \xi)|_{\Omega(f)}\big) = \{\mu_{\mathcal{O}}\}.$$

Hence, $\phi+\widetilde{\psi} \in \operatorname{Lock}\big(f, C^{0,\alpha}(M,\mathbb{R})\big)$, which implies $\operatorname{Lock}\big(f, C^{0,\alpha}(M,\mathbb{R})\big)$ is dense in $C^{0,\alpha}(M,\mathbb{R})$. □

Proof of Theorem A. Suppose $r \in \mathbb{N}$, and $\mathcal{P}$ is either $C^1(M,\mathbb{R})$ or $C^{0,\alpha}(M,\mathbb{R})$. By Proposition 3.6, $\operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$ is dense in $\mathcal{A}^r(M) \times \mathcal{P}$. So it suffices to prove that $\operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$ is open.

Note that every $\phi \in \mathcal{P}$ that appears in a pair $(f,\phi) \in \operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$ must be nonconstant. Indeed, if ϕ is a constant function, then every f-invariant measure attains the maximum ergodic average $Q(f,\phi)$, so $\mathcal{M}_{\max}(f,\phi) = \mathcal{M}(M,f)$; in particular the maximizing measure is not unique, contradicting the PO property. Hence no constant function ϕ can belong to $\operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$.

Now fix $(f,\phi) \in \operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$; by the above observation, ϕ is nonconstant. By Lemmas 3.2, 3.3, and 3.5, f is $\mathcal{A}^r(M)$-stably hyperbolic. Since ϕ is not a constant, Theorem 2.5 implies that (f,ϕ) is in the interior of $\operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$. Therefore, $\operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P})$ is an open subset of $\mathcal{A}^r(M) \times \mathcal{P}$. □

Remark 3.7. Lemma 3.2 provides the stronger conclusion that h_g can be taken to be a homeomorphism for g sufficiently close to f, but this additional property is not needed for the proof of Theorem A itself. The abstract framework of Section 2 requires only the intertwining properties of h_g and i_g from condition (IS) in Definition 2.1, and these alone suffice to establish that $\{(f,\phi) \in \operatorname{Lock}(\mathcal{A}^r(M),\mathcal{P}) : \phi \text{ is nonconstant}\}$ is open and dense in $\mathcal{A}^r(M) \times \mathcal{P}$.

4. Rational maps on the Riemann sphere: proof of Theorems B and C

Let $|\cdot|$ be a conformal metric on $\widehat{\mathbb{C}}$ with induced distance function d. Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map with $\deg f \geqslant 2$, and denote the Julia set of f by $J(f)$. An f-periodic point x of period $p \in \mathbb{N}$, and its orbit, are called *attracting* (resp. *repelling*) if $|(f^p)'(x)| < 1$ (resp. $|(f^p)'(x)| > 1$), and *hyperbolic* if x is either attracting or repelling. Let $AP(f)$ denote the set of all attracting periodic points of f. For an attracting periodic orbit $\mathcal{O}$, the *attracting basin* of $\mathcal{O}$ is defined as $B_f(\mathcal{O}) \coloneqq \big\{x \in \widehat{\mathbb{C}} : \lim_{n\to+\infty} d(f^n(x),\mathcal{O}) = 0\big\}$.

The following lemma is a standard result:

Lemma 4.1. *If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a hyperbolic rational map with $\deg f \geqslant 2$, then its nonwandering set $\Omega(f)$ is equal to the disjoint union of its Julia set and its finitely many attracting periodic orbits, and also equal to the closure of the set of f-periodic points.*

Proof. By [Mi06, Theorem 19.1], every orbit in the Fatou set of f converges to an attracting periodic orbit, so $\Omega(f) \subseteq J(f) \cup AP(f)$. On the other hand, by [Mi06, Theorem 14.1], the set of repelling periodic points is dense in $J(f)$, so the set of f-periodic points is dense in $J(f) \cup AP(f)$, and every periodic point is nonwandering, so $J(f) \cup AP(f) \subseteq \Omega(f)$. Since $f|_{J(f)}$ is expanding, $J(f) \cap AP(f) = \emptyset$, so indeed $\Omega(f)$ is the disjoint union of $J(f)$ and $AP(f)$, and the f-periodic points are dense in $\Omega(f)$. The number of attracting periodic points is finite (see e.g. [Mi06, Theorem 8.2]), so the result is proved. □

We next prove, by means of the following Lemmas 4.2, 4.3, 4.4, and 4.5, that every hyperbolic rational map of degree $m \geqslant 2$ is $\mathcal{HR}^m$-stably hyperbolic. We first prove the following enhanced version of condition (IS) in Definition 2.1:

Lemma 4.2. *Suppose $f \in \mathcal{HR}^m$ for some $m \geqslant 2$. For each $\varepsilon > 0$, and for each $g \in \mathcal{HR}^m$ sufficiently close to f, there exists a homeomorphism $h_g\colon \Omega(f) \to \Omega(g)$ with $d_\infty(h_g, \mathrm{id}) < \varepsilon$ and $h_g \circ f = g \circ h_g$. Moreover, $h_g(J(f)) = J(g)$ and $h_g(AP(f)) = AP(g)$.*

Proof. Note that $\mathcal{HR}^m$ is an open set ([Mi06, p. 205]). So rational maps of degree m in a neighbourhood of f are hyperbolic, so each of their periodic points is hyperbolic. For any rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, denote the set of g-periodic points by $P(g)$. By Lemma 4.1, $\Omega(g)$ is equal to the closure of $P(g)$. Thus, by [Ly83, Main Lemma] (see also [MSS83] or [MS98]), for $g \in \mathcal{HR}^m$ sufficiently close to f, there exists a topological conjugacy $h_g\colon \Omega(f) \to \Omega(g)$. Furthermore (see [Ly83, Main Lemma] and [Ly86, Section 2.1]), h_g converges to the identity uniformly as g converges to f. If $g \in \mathcal{HR}^m$ and the conjugacy h_g exists, by [Mi06, Corollary 4.14], $J(f)$ and $J(g)$ have no isolated points. But every point in $AP(f)$ is isolated in $\Omega(f)$, and every point in $AP(g)$ is isolated in $\Omega(g)$. Now $h_g\colon \Omega(f) \to \Omega(g)$ is a homeomorphism, so $h_g(J(f)) = J(g)$ and $h_g(AP(f)) = AP(g)$, and the result is proved. □

Recall that if (X, d) is a compact metric space and $T\colon X \to X$, then T is *distance-expanding* if there exist constants $\eta > 0$ and $\theta > 1$ such that for any two distinct points x, $y \in X$, if $d(x, y) < \eta$ then $d(T(x), T(y)) \geqslant \theta d(x, y)$. We shall refer to (η, θ) as a pair of *expanding constants* for T.

Lemma 4.3. *Suppose $f \in \mathcal{HR}^m$ for some $m \geqslant 2$. Equip $\widehat{\mathbb{C}}$ with a conformal metric $|\cdot|$ satisfying (1.4) for some $\lambda > 1$, and let d denote its induced distance. Then the restriction $f|_{J(f)}$ is an open mapping, and is distance-expanding. Moreover, the expanding constants can be chosen uniformly over all rational maps in some neighbourhood of f. More precisely, there exist a neighbourhood $N \subseteq \mathcal{HR}^m$ of f, and constants $\eta > 0$, $\theta > 1$, such that for all $g \in N$ and distinct points x, $y \in J(g)$, if $d(x, y) < \eta$, then $d(g(x), g(y)) \geqslant \theta d(x, y)$.*

Proof. See Appendix A. □

Lemma 4.3 allows us to verify condition (RHE) in Definition 2.1:

Lemma 4.4. *If $f \in \mathcal{HR}^m$ for some $m \geqslant 2$, then f satisfies condition (RHE) in Definition 2.1 for $\mathcal{F} \coloneqq \mathcal{HR}^m$.*

Proof. Let $\eta > 0, \theta > 1$ be the constants, and N the neighbourhood, that are guaranteed by Lemma 4.3. Now $J(f)$ and $AP(f)$ are disjoint compact sets, and $AP(f)$ is finite, so $d(J(f), AP(f)) > 0$, and $\Delta(AP(f))$ is well defined. Define

$$\delta \coloneqq \min\bigl\{3^{-1} d(J(f), AP(f)),\, 3^{-1}\Delta(AP(f)),\, \eta\bigr\}. \tag{4.1}$$

By Lemma 4.2, there exists a neighbourhood U_0 of f such that if $g \in U_0$ then h_g exists and $d_\infty(h_g, \mathrm{id}) < \delta$. Thus, by (4.1), for every $g \in U_0$, we have $d(J(g), AP(g)) > d(J(f), AP(f)) - 2\delta \geqslant \delta$ and $\Delta(AP(g)) \geqslant \Delta(AP(f)) - 2\delta \geqslant \delta$. This means that if $g \in U_0$ and x, $y \in \Omega(g)$ with $d(x, y) < \delta$, then x, $y \in J(g)$. Define $U \coloneqq N \cap U_0$. Fix $g \in U$, $n \in \mathbb{N}$, and x, $y \in \Omega(g)$ with $\max\bigl\{d\bigl(g^i(x), g^i(y)\bigr) : 0 \leqslant i \leqslant n\bigr\} < \delta$. Then $\bigl\{x,\, y,\, g^1(x),\, g^1(y),\, \dots,\, g^n(x),\, g^n(y)\bigr\} \subseteq J(g)$. Moreover, by Lemma 4.3, we have $d\bigl(g^i(x), g^i(y)\bigr) \leqslant \theta^{i-n} d(g^n(x), g^n(y))$, which in particular implies that condition (RHE) in Definition 2.1 holds with $K \coloneqq 1$ and $\lambda \coloneqq \theta$. □

We now verify condition (ML) in Definition 2.1:

Lemma 4.5. *If $f \in \mathcal{HR}^m$ for some $m \geqslant 2$, then f satisfies condition (ML) in Definition 2.1 for $\mathcal{F} \coloneqq \mathcal{HR}^m$.*

Proof. The strategy of proof will be that for all $\phi \in C^{0,\alpha}\bigl(\widehat{\mathbb{C}}\bigr)$, the function $u = u_\phi$ can be constructed separately on $J(f)$ and $AP(f)$.

First we restrict f to the Julia set $J(f)$. By Lemma 4.3, $f|_{J(f)}$ is open and distance-expanding, so by [LS26, Proposition 3.6] there exists $L_J > 0$ such that for all $\phi \in C^{0,\alpha}\bigl(\widehat{\mathbb{C}}\bigr)$, there exists $v_\phi \in C^{0,\alpha}(J(f))$ satisfying $|v_\phi|_{\alpha,J(f)} \leqslant L_J |\phi|_{\alpha,J(f)}$ and $\overline{\phi} + v_\phi - v_\phi \circ f \leqslant 0$ on $J(f)$. By adjusting v_ϕ by an additive constant, we may assume that $v_\phi(x) = 0$ for some $x \in J(f)$, which implies that

$$\|v_\phi\|_{\infty,J(f)} \leqslant \bigl(\operatorname{diam}\widehat{\mathbb{C}}\bigr)^\alpha |v_\phi|_{\alpha,J(f)} \leqslant L_J \bigl(\operatorname{diam}\widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha,J(f)} \leqslant L_J \bigl(\operatorname{diam}\widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha,\widehat{\mathbb{C}}}. \tag{4.2}$$

Next we restrict f to $AP(f)$, and define a function $w_\phi\colon AP(f) \to \mathbb{R}$ as follows. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the finitely many attracting f-periodic orbits. Define $p_i := \operatorname{card} \mathcal{O}_i$ for each $1 \leqslant i \leqslant k$, and $p := \max_{1 \leqslant i \leqslant k} p_i$. For each integer $1 \leqslant i \leqslant k$, choose a point $x \in \mathcal{O}_i$ and define

$$w_\phi\bigl(f^j(x)\bigr) := S_j^f \overline{\phi}(x) \quad \text{ for each } 0 \leqslant j \leqslant p_i - 1.$$

By our construction, if $0 \leqslant j \leqslant p_i - 2$, then $\overline{\phi}\bigl(f^j(x)\bigr) + w_\phi\bigl(f^j(x)\bigr) - w_\phi\bigl(f^{j+1}(x)\bigr) = 0$, and $\overline{\phi}\bigl(f^{p_i-1}(x)\bigr) + w_\phi\bigl(f^{p_i-1}(x)\bigr) - w_\phi\bigl(f^{p_i}(x)\bigr) = S_{p_i}^f \overline{\phi}(x) \leqslant 0$.

So we conclude that $\overline{\phi} + w_\phi - w_\phi \circ f \leqslant 0$ on $AP(f)$. Moreover, by our construction, we have $\|w_\phi\|_{\infty, AP(f)} \leqslant p \|\overline{\phi}\|_{\infty, \widehat{\mathbb{C}}}$. Note that $Q\bigl(f, \overline{\phi}\bigr) = 0$, so $\overline{\phi}(x) \geqslant 0$ at some $x \in \widehat{\mathbb{C}}$ and $\overline{\phi}(y) \leqslant 0$ for some $y \in \widehat{\mathbb{C}}$, which implies that $\|\overline{\phi}\|_{\infty, \widehat{\mathbb{C}}} \leqslant \bigl(\operatorname{diam} \widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha, \widehat{\mathbb{C}}}$. Therefore,

$$\|w_\phi\|_{\infty, AP(f)} \leqslant p \|\overline{\phi}\|_{\infty, \widehat{\mathbb{C}}} \leqslant p \bigl(\operatorname{diam} \widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha, \widehat{\mathbb{C}}}. \tag{4.3}$$

Let $\tau := \min\{d(J(f), AP(f)), \Delta(AP(f))\}$ and then define the function $u_\phi\colon \Omega(f) \to \mathbb{R}$ by $u_\phi := v_\phi$ on $J(f)$ and $u_\phi := w_\phi$ on $AP(f)$. Fix distinct points $x, y \in \Omega(f)$. When $\{x, y\} \subseteq AP(f)$, combining the fact that $d(x,y) \geqslant \tau$ and (4.3) gives $|u_\phi(x) - u_\phi(y)|/d(x,y)^\alpha \leqslant 2p\tau^{-\alpha} \bigl(\operatorname{diam} \widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha, \widehat{\mathbb{C}}}$. When either $x \in AP(f)$ and $y \in J(f)$, or $x \in J(f)$ and $y \in AP(f)$, combining the fact that $d(x,y) \geqslant \tau$, (4.2), and (4.3) gives $|u_\phi(x) - u_\phi(y)|/d(x,y)^\alpha \leqslant (p + L_J)\tau^{-\alpha} \bigl(\operatorname{diam} \widehat{\mathbb{C}}\bigr)^\alpha |\phi|_{\alpha, \widehat{\mathbb{C}}}$. When $\{x, y\} \subseteq J(f)$, then $|u_\phi(x) - u_\phi(y)|/d(x,y)^\alpha \leqslant L_J |\phi|_{\alpha, \widehat{\mathbb{C}}}$. From these bounds we see that f satisfies condition (ML) in Definition 2.1 with $L := \max\bigl\{L_J, \, (2p + L_J)\tau^{-\alpha} \bigl(\operatorname{diam} \widehat{\mathbb{C}}\bigr)^\alpha\bigr\}$. □

The final ingredient needed for the proofs of Theorems B and C is the Individual TPO for hyperbolic rational maps.

Lemma 4.6. *Let $\alpha \in (0,1]$, $f \in \mathcal{HR}^m$ for some $m \geqslant 2$, and $\mathcal{P}$ denote either $C^{0,\alpha}\bigl(\widehat{\mathbb{C}}\bigr)$ or $C^1\bigl(\widehat{\mathbb{C}}\bigr)$. Then* $\operatorname{Lock}(f, \mathcal{P})$ *is an open dense subset of $\mathcal{P}$.*

Proof. Provided $(\Omega(f), f)$ satisfies conditions (ACP), (EI), and (NLP) from [HLMXZ25], the result follows by an argument analogous to the one used to prove Theorem 3.6.

To check that indeed $(\Omega(f), f)$ does satisfy these three conditions, we note that (EI) and (NLP) follow from Lemmas 4.4 and 4.5, respectively. For the condition (ACP), if

$$0 < \varepsilon < \min\{d(J(f), AP(f)), \, \Delta(AP(f))\},$$

then since $f(J(f)) = J(f)$ and $f(AP(f)) = AP(f)$, every ε-pseudo orbit is either contained entirely in $J(f)$, or contained entirely within one of the attracting periodic orbits that makes up $AP(f)$. For pseudo orbits contained in a periodic orbit, property (ACP) is trivially satisfied. For pseudo orbits contained in $J(f)$, property (ACP) follows from [PU10, Proposition 4.2.3], since $(J(f), f)$ is open and distance-expanding by Lemma 4.3. □

Finally, we can prove Theorems B and C.

Proof of Theorem B. This follows by combining Lemmas 4.2, 4.4, 4.5, 4.6, and Theorem 2.5, and using an argument analogous to the one used to prove Theorem A. □

Proof of Theorem C. Let $\alpha \in (0,1]$, and let $\mathcal{P}$ denote either $C^1\bigl(\widehat{\mathbb{C}}\bigr)$ or $C^{0,\alpha}\bigl(\widehat{\mathbb{C}}\bigr)$. Let $\mathcal{HQ}$ denote the set of hyperbolic real quadratic polynomials. Note that $\mathcal{Q}$ and $\mathcal{HR}^2$ are both equipped with the standard topology on the parameter space. Since $\mathcal{HQ} = \mathcal{HR}^2 \cap \mathcal{Q}$ and $\mathcal{HR}^2$ is open (see [Mi06, p. 205]), it follows that $\mathcal{HQ}$ is open. By [GS97] and [Ly97, p. 4], $\mathcal{HQ}$ is a dense subset of $\mathcal{Q}$, so it suffices to prove the Joint TPO property for $\mathcal{HQ} \times \mathcal{P}$.

By Lemmas 4.2, 4.4, and 4.5, every $f \in \mathcal{HQ} \subseteq \mathcal{HR}^2$ is stably hyperbolic (i.e., satisfies Definition 2.1) for $\mathcal{F} := \mathcal{HR}^2$, and hence also for $\mathcal{F} := \mathcal{HQ}$. By this and Theorem 2.5, the set $\mathcal{L}_0 := \{(f, \phi) \in \operatorname{Lock}(\mathcal{HQ}, \mathcal{P}) : \phi \text{ is not a constant}\}$ is an open subset of $\mathcal{HQ} \times \mathcal{P}$. Since $\operatorname{Const} := \mathcal{HQ} \times$

$\{\phi \in \mathcal{P} : \phi \text{ is a constant}\}$ is nowhere dense in $\mathcal{HQ} \times \mathcal{P}$, by Lemma 4.6, $\mathcal{L}_0 = \mathrm{Lock}(\mathcal{HQ}, \mathcal{P}) \setminus \mathrm{Const}$ is dense in $\mathcal{HQ} \times \mathcal{P}$, so the result is proved. □

5. C^r one-dimensional maps and the logistic family: proof of Theorems D and E

We first recall some basic definitions and properties in one-dimensional dynamics (see [dMvS93, Chapter 3] for more background). We follow the conventions in [dMvS93] and [KSS07].

Let $r \in \mathbb{N}$, M be a compact interval or a circle equipped with the Euclidean metric, and $f \in C^r(M, M)$. A subset $K \subseteq M$ is said to be a *hyperbolic* set for f if $f(K) \subseteq K$ and there exist constants $C > 0$ and $\theta > 1$ such that $|(f^n)'(x)| \geqslant C\theta^n$ for all $x \in K$ and $n \in \mathbb{N}$. It can be seen from [dMvS93, p. 226] that there exist a Riemannian metric $|\cdot|_f$ and a constant $\lambda > 1$ such that

$$|\mathrm{D}f|_f > \lambda \quad \text{on } K(f). \tag{5.1}$$

Since M is compact, any two Riemannian metrics are mutually equivalent, so the topology of $C^r(M, \mathbb{R})$ and $C^r(M, M)$ are independent of the choice of metric.

An f-periodic point x, of period p, is said to be *attracting* if $|(f^p)'(x)| < 1$, and the orbit of an attracting periodic point is called an attracting periodic orbit. As in the case of rational maps, if $x \in M$ is an attracting periodic point, then so is every $y \in \mathcal{O}^f(x)$. The map f is said to be *hyperbolic* (or, as alternative terminology, to satisfy *Axiom A*) if f has a hyperbolic set $K(f)$ and finitely many attracting periodic points such that the forward orbit under f at every point in $M \setminus K(f)$ converges to an attracting periodic orbit (cf. [dMvS93, p. 221]).

Kozlovski, Shen & van Strien [KSS07, Theorem 2] showed that hyperbolic maps are dense in $C^r(M, M)$, for every $r \in \mathbb{N}$. We will use their result, together with our theorems from Section 2, to prove Theorem D.

We first restrict our arguments to those maps in $C^r(M, M)$ that preserve the endpoints, in the case that M is an interval. More precisely, we denote $C^r_0(M, M) := C^r(M, M)$ when M is the circle, and $C^r_0(M, M) := \{f \in C^r(M, M) : f(\partial M) \subseteq \partial M\}$ when M is a compact interval. Let $HC^r_0(M, M)$ denote the space of hyperbolic maps in $C^r_0(M, M)$.

For such maps, we have:

Proposition 5.1. *Let $r \in \mathbb{N}$, $\alpha \in (0, 1]$, M be a compact interval or a circle, and $\mathcal{P}$ denote $C^{0,\alpha}(M, \mathbb{R})$ or $\mathcal{P} := C^1(M, \mathbb{R})$. The space $HC^r_0(M, M)$ is an open subset of $C^r_0(M, M)$. If $f \in HC^r_0(M, M)$, then f is $C^r_0(M, M)$-stably hyperbolic in the sense of Remark 2.6 (i), and if $\phi \in \mathrm{Lock}(f, \mathcal{P})$ is not a constant, then (f, ϕ) is in the interior of $\mathrm{Lock}(C^r_0(M, M), \mathcal{P})$.*

Proof. Denote by $HC^r_0(M, M)$ the set of hyperbolic maps in $C^r_0(M, M)$. It follows from [dMvS93, Theorem 2.4, Chapter 3] that $HC^r_0(M, M)$ is an open subset of $C^r_0(M, M)$.

For each $g \in HC^r_0(M, M)$, define $L(g)$ to be the union of $K(g)$ and the set of all attracting periodic points of g. By definition, we have $g(L(g)) \subseteq L(g)$. Since the forward orbit of every point outside $L(g)$ converges to an attracting periodic orbit, it follows that $\Omega(g) \subseteq L(g)$. Consequently, every g-invariant measure is supported on $L(g)$.

Replacing $\Omega(g)$ by $L(g)$ in Definition 2.1, it follows from Theorem 2.5, Remark 2.6 (i), and the openness of $HC^r_0(M, M)$ in $C^r_0(M, M)$, that it suffices to verify that every hyperbolic map $g \in C^r_0(M, M)$ is $HC^r_0(M, M)$-stably hyperbolic.

Condition (IS) in Definition 2.1 for each $f \in HC^r_0(M, M)$ follows from [dMvS93, Theorem 2.4, Chapter 3]. Indeed, this result states that if $f \in C^r_0(M, M)$ is hyperbolic, then for every $g \in C^r_0(M, M)$ sufficiently close to f, there exists a conjugacy $h_g : K(f) \to K(g)$ that converges uniformly to the identity as g tends to f. Moreover, an inspection of the proof of [dMvS93, Theorem 2.4, Chapter 3] shows that h_g can be extended to a conjugacy $i_g : L(f) \to L(g)$ that also converges uniformly to the identity as g tends to f.

For conditions (RHE) and (ML) in Definition 2.1, given $f \in HC^r_0(M,M)$, consider the metric $|\cdot|_f$ and a constant $\lambda > 1$ such that $|\mathrm{D}f|_f > \lambda$ on $K(f)$. The verification then proceeds exactly as in the proofs of Lemmas 4.3, 4.4, and 4.5 if we replace $J(f)$ by $K(f)$. □

Lemma 5.2. *Let $r \in \mathbb{N}$, $\alpha \in (0,1]$, M be a circle or a compact interval, $f \in C^r_0(M,M)$ be a hyperbolic map, and $\mathcal{P}$ denote $C^1(M,\mathbb{R})$ or $C^{0,\alpha}(M,\mathbb{R})$. Then* $\mathrm{Lock}(f,\mathcal{P})$ *is an open dense subset of $\mathcal{P}$.*

Proof. Arguing as in Lemma 4.3, by equipping M with the metric $|\cdot|_f$, it can be shown that $f|_{K(f)}$ is distance-expanding. Moreover, $|\mathrm{D}f|_f$ does not vanish on a neighbourhood V of $K(f)$, which implies that $f|_V$ is an open map. Since the forward orbit of every point outside $K(f)$ converges to an attracting periodic orbit, it follows that $f^{-1}(K(f)) = K(f)$. Hence, $f|_{K(f)}$ is also an open map.

Using an argument analogous to that in Lemma 4.6, one verifies that f satisfies the (ACP) property in [HLMXZ25]. Furthermore, conditions (RHE) and (ML) in Definition 2.1 established in Proposition 5.1 imply that f also satisfies the properties (EI) and (NLP) of [HLMXZ25]. The conclusion then follows by an argument analogous to that of Proposition 3.6. □

Applying Proposition 5.1 and Lemma 5.2 to Theorem D requires a transition from $C^r(M,M)$ to the subclass $C^r_0(M,M)$. We facilitate this by developing a method to extend general interval maps to endpoint-preserving ones without altering their fundamental dynamical properties. The details of this construction follow.

Lemma 5.3. *Let $M = [a,b]$ be an interval, $r \in \mathbb{N}$, and $f \in C^r(M,M)$. Assume that $\{f(a),\, f(b)\} \subseteq (a,b)$ and $f'(a) \neq 0 \neq f'(b)$. Denote $\tau := \frac{1}{2}\min\{f(a) - a,\, b - f(b)\} > 0$. Then there exists $a_0 < a$, $b_0 > b$, and $F \in C^r_0([a_0,b_0],[a_0,b_0])$ satisfying the following properties:*

(i) *$\{F(a_0),\, F(b_0)\} \subseteq \{a_0,\, b_0\}$ and $F|_M = f$.*
(ii) *For every $x \in [a_0,a] \cup [b,b_0]$, $F'(x) \neq 0$.*
(iii) *There exist $\theta > 1$, $a_1 \in (a_0,a)$, and $b_1 \in (b,b_0)$ such that $F([a_0,a_1) \cup (b_1,b_0]) \cap [a+\tau, b-\tau] = \emptyset$ and $F([a_1,a] \cup [b,b_1]) \subseteq [a+\tau, b-\tau]$, and $|F'| > \theta$ on $[a_0,a_1) \cup (b_1,b_0]$.*

Lemma 5.4. *Let the intervals $M = [a,b]$ and $M_0 = [a_0,b_0]$ be such that $a_0 < a < b < b_0$. Let $r \in \mathbb{N}$. Then there exists a constant $D_r \geqslant 1$ such that for every $f \in C^r(M,\mathbb{R})$, there exists a C^r extension F of f to M_0 with $F(a_0) = F(b_0) = 0$ and $\|F\|_{C^r,M_0} \leqslant D_r \|f\|_{C^r,M}$.*

The proofs of Lemmas 5.3 and 5.4 rely on some auxiliary technical arguments unconnected to the dynamical ideas used in the main proof of Theorem D, so are relegated to Appendix A.

Lemma 5.5. *Let $M = [a,b]$ be an interval, $r \in \mathbb{N}$, and $f \in C^r(M,M)$, and let $M_0 := [a_0,b_0]$ and F be as in Lemma 5.3. Then F satisfies the following properties:*

(i) *There exists $\kappa > 0$ such that for each $H \in C^r(M_0,\mathbb{R})$ with $H(a_0) = H(b_0) = 0$ and $\|H\|_{C^r,M_0} < \kappa$, we have $F + H \in C^r_0(M_0,M_0)$.*
(ii) *There exists a neighbourhood $W \subseteq C^r_0(M_0,M_0)$ of F such that for every $G \in W$ with $G(M) \subseteq M$, every G-invariant measure is supported on $M \cup \{a_0,\, b_0\}$.*
(iii) *F is hyperbolic.*
(iv) *If W is the neighbourhood of F as in (ii), then for every $G \in W$ with $G(M) \subseteq M$, if $\phi \in C(M_0)$ is such that $\phi(a_0) < Q(G,\phi)$ and $\phi(b_0) < Q(G,\phi)$, then every (G,ϕ)-maximizing measure has support contained in M.*

Proof. Our proof will use the properties of F from Lemma 5.3, where the constants τ and θ are as in that lemma.

(i) Define $\kappa := \min\{\theta,\, a-a_0,\, b_0-b\}$. Clearly, $\kappa > 0$. Let $H \in C^r(M_0,\mathbb{R})$ satisfy $H(a_0) = H(b_0) = 0$ and $\|H\|_{C^r,M_0} < \kappa$. Set $\widetilde{F} := F + H$. Then $\{\widetilde{F}(a_0),\, \widetilde{F}(b_0)\} \subseteq \{a_0,\, b_0\}$. Lemma 5.3 (iii) implies that $a \leqslant F \leqslant b$ on $[a_1, b_1]$. Hence

$$a_0 \leqslant a - \kappa < \widetilde{F} < b + \kappa \leqslant b_0 \quad \text{ on } [a_1, b_1],$$

which yields $\widetilde{F}([a_1, b_1]) \subseteq M_0$. Moreover, Lemma 5.3 (iii) also gives $|\widetilde{F}'| \geqslant |F'| - |H'| > \theta - \kappa \geqslant 0$ on $[a_0, a_1)$, so $\widetilde{F}$ is monotonic. Since $\widetilde{F}(a_0) = F(a_0) \in M_0$ and $\widetilde{F}(a_1) \in M \subseteq M_0$, then $\widetilde{F}([a_0, a_1)) \subseteq M_0$. Similarly, $\widetilde{F}((b_1, b_0]) \subseteq M_0$.

In conclusion, $\widetilde{F} \in C^r(M_0,\mathbb{R})$, $\{\widetilde{F}(a_0),\, \widetilde{F}(b_0)\} \subseteq \{a_0,\, b_0\}$, $\widetilde{F}([a_0, b_0]) \subseteq [a_0, b_0]$, and therefore $\widetilde{F} \in C^r_0(M_0, M_0)$.

(ii) Define $\eta := \min\{\tau,\, 2^{-1}(\theta-1)\}$ and $W := B_{C^r_0(M_0,M_0)}(F, \eta)$, where $B_{C^r_0(M_0,M_0)}(F, \eta)$ denotes the set of functions $G \in C^r_0(M_0, M_0)$ with $\|G - F\|_{C^r,M_0} < \eta$.

Fix $G \in W$ with $G(M) \subseteq M$, and set $M_1 := M \cup \{a_0,\, b_0\}$. Since $G \in C^r_0(M_0, M_0)$ and $G(M) \subseteq M$, we have $G(M_1) \subseteq M_1$. To show that every G-invariant probability measure is supported on M_1, it suffices to show that for every $x \in M_0 \smallsetminus M_1 = (a_0, a) \cup (b, b_0)$, there exists $n \in \mathbb{N}$ such that $G^n(x) \in M_1$. Indeed, if this holds, the forward invariance $G(M_1) \subseteq M_1$ ensures that the forward orbit of x never returns to the open set $M_0 \smallsetminus M_1$, meaning no point in $M_0 \smallsetminus M_1$ is recurrent. By the Poincaré recurrence theorem, any G-invariant measure must assign zero mass to $M_0 \smallsetminus M_1$, and therefore has support contained in M_1.

We now prove this forward orbit property. First consider the case where $x \in (a_0, a)$. Suppose for a contradiction that for every $n \in \mathbb{N}$, $G^n(x) \notin M_1$. By Lemma 5.3 (iii), $F([a_1, a] \cup [b, b_1]) \subseteq [a+\tau, b-\tau]$. Since $\|G - F\|_{\infty,M_0} < \eta \leqslant \tau$, it follows that $G([a_1, a] \cup [b, b_1]) \subseteq M$. Hence $G^n(x) \notin [a_1, b_1]$ for all $n \in \mathbb{N}$.

Now since $\|F' - G'\|_{\infty,M_0} < \eta \leqslant 2^{-1}(\theta - 1)$ and $|F'| > \theta$ on $(a_0, a_1) \cup (b_1, b_0)$, again by Lemma 5.3 (iii), we obtain that

$$|G'| \geqslant |F'| - \|F' - G'\|_{\infty,M_0} > 2^{-1}(\theta + 1) > 1 \quad \text{ on } (a_0, a_1) \cup (b_1, b_0).$$

So G is monotone on both (a_0, a_1) and (b_1, b_0). As we have shown that $G(a_1)$, $G(b_1) \in M$, we have

$$\begin{aligned} &\text{either} \quad G([a_0, a_1]) \subseteq [a_0, b] \quad \text{or} \quad G([a_0, a_1]) \subseteq [a, b_0], \\ &\text{either} \quad G([b_1, b_0]) \subseteq [a_0, b] \quad \text{or} \quad G([b_1, b_0]) \subseteq [a, b_0]. \end{aligned} \tag{5.2}$$

Since $|G'| > 2^{-1}(1 + \theta)$ on $[a_0, x]$ and $G(x) \notin [a_1, b_1]$, the image $G([a_0, x])$ is a closed interval with one endpoint in $\{a_0, b_0\}$ and length $|G([a_0, x])| > 2^{-1}(1 + \theta)(x - a_0)$. Thus, because $G([a_0, a_1])$ cannot cover both (a_0, a_1) and (b_1, b_0) simultaneously without violating (5.2), the fact that $G(x) \notin [a_1, b_1]$ forces $G([a_0, x])$ to be contained entirely in $[a_0, a_1)$ or entirely in $(b_1, b_0]$. In particular, $|G'| > 2^{-1}(1 + \theta)$ on all of $G([a_0, x])$. Using (5.2) and the fact that $G^n(x) \notin [a_1, b_1]$ for all $n \in \mathbb{N}$, we may iterate the arguments above to obtain

$$|G^n([a_0, x])| > 2^{-n}(1 + \theta)^n(x - a_0) \quad \text{ for all } n \in \mathbb{N}.$$

However, $\lim_{n\to+\infty} 2^{-n}(1+\theta)^n(x-a_0) = +\infty$, which is impossible since $G^n([a_0, x]) \subseteq [a_0, b_0]$. This contradiction proves that there exists $n \in \mathbb{N}$ such that $G^n(x) \in M_1$.

The proof in the case that $x \in (b, b_0)$ is entirely analogous to the one above for the case that $x \in (a_0, a)$, and will be omitted. Therefore, (ii) is proved.

(iii) Set $K_0 := \{a_0,\, b_0\} \cup \bigcup_{i=0}^{+\infty} F^{-i}(K(f))$. We next show that F is hyperbolic for $K(F) = K_0$.

Since $F(\{a_0,\, b_0\}) \subseteq \{a_0,\, b_0\}$, by the definition of K_0 we have $F(K_0) \subseteq K_0$. Fix $x \in [a_0, b_0] \smallsetminus K_0$. If $x \in M$, then $x \in M \smallsetminus K(f)$. By the hyperbolicity of f, the orbit $\{F^i(x)\}_{i\in\mathbb{N}}$ converges to an attracting periodic orbit of f, which is also an attracting periodic orbit of F.

If $x \notin M$, then $x \in (a_0, a) \cup (b, b_0)$. By (ii), there exists $n \in \mathbb{N}$ such that $F^n(x) \in M$. Since $x \notin K_0$, we have $x \notin F^{-n}(K(f))$, and hence $F^n(x) \in M \smallsetminus K(f)$. Consequently, the orbit $\{F^{n+i}(x)\}_{i\in\mathbb{N}}$

converges to an attracting periodic orbit of F. In conclusion, for every point $x \in M_0 \smallsetminus K_0$, the orbit $\{F^i(x)\}_{i\in\mathbb{N}}$ converges to an attracting periodic orbit of F.

Next, we prove that F is expanding on K_0. By the hyperbolicity of f, there exist constants $C > 0$ and $\lambda > 1$ such that $|(f^m)'(y)| \geqslant C\lambda^m$ for all $y \in K(f)$ and $m \in \mathbb{N}$. We claim that $f^{-n}(K(f)) \subseteq K(f)$ for all $n \in \mathbb{N}$. Otherwise, there would exist $n \in \mathbb{N}$ and $x \in f^{-n}(K(f)) \smallsetminus K(f)$. Then $\{f^i(x)\}_{i\in\mathbb{N}}$ converges to an attracting periodic orbit of f, which contradicts the fact that $|(f^m)'(f^n(x))| \geqslant C\lambda^m$ for every $m \in \mathbb{N}$. Using this, we obtain $M \cap K_0 = K(f)$. Indeed, clearly, $K(f) \subseteq M \cap K_0$. Conversely, if $y \in K_0 \cap M$, then there exists $n \in \mathbb{N}_0$ such that $f^n(y) = F^n(y) \in K(f)$. Since $f^{-n}(K(f)) \subseteq K(f)$, we conclude that $y \in K(f)$.

Fix $x \in K_0$. If $x \in \{a_0, b_0\}$, by Lemma 5.3 (i)(iii), we have $F(\{a_0, b_0\}) \subseteq \{a_0, b_0\}$, $|F'(a_0)| > \theta$, and $|F'(b_0)| > \theta$. Thus, for every $m \in \mathbb{N}$, we have $|(F^m)'(a_0)| > \theta^m$ and $|(F^m)'(b_0)| > \theta^m$. If $x \in M$, then $x \in K(f)$, and hence $|(F^m)'(x)| \geqslant C\lambda^m$ for every $m \in \mathbb{N}$. If $x \in (a_0, a) \cup (b, b_0)$, by (ii) there exists $k \in \mathbb{N}$ such that $F^k(x) \in M$ and $F^{k-1}(x) \notin M$. Then, $F^k(x) \in K(f)$. By Lemma 5.3 (iii), we have $F([a_1, b_1]) \subseteq M$, and hence $F^i(x) \notin [a_1, b_1]$ for all $i \in \{0, \dots, k-2\}$. So Lemma 5.3 (iii) implies that for all $i \in \{0, \dots, k-2\}$, $\left|F'\left(F^i(x)\right)\right| > \theta$. Moreover by Lemma 5.3 (ii), the constant $s := \min\{|F'(y)| : y \in [a_0, a] \cup [b, b_0]\} > 0$ is well defined. Thus, $|(F^m)'(x)| \geqslant s\theta^{m-1}$ if $1 \leqslant m \leqslant k$ and $|(F^m)'(x)| \geqslant Cs\theta^{k-1}\lambda^{m-k}$ if $m \geqslant k+1$. Consequently, in all cases, we have $|(F^m)'(x)| \geqslant \min\{1, C, s/\theta, Cs/\theta\} \cdot \min\{\theta, \lambda\}^m$ for every $m \in \mathbb{N}$. Hence, F is expanding on K_0.

It remains to prove that K_0 is compact. Since $F^{-i}(K(f))$ is compact for every $i \in \mathbb{N}_0$, it suffices to show that for any sequence $\{x_i\}_{i\in\mathbb{N}_0}$ satisfying $x_i \in F^{-i}(K(f)) \smallsetminus \bigcup_{j=0}^{i-1} F^{-j}(K(f))$ for each $i \in \mathbb{N}_0$, every limit point of $\{x_i\}_{i\in\mathbb{N}_0}$ belongs to K_0. Let $\{x_i\}_{i\in\mathbb{N}_0}$ be such a sequence and let x_* be a limit point. We claim that $x_* \in \{a_0, b_0\}$.

Fix $i \in \mathbb{N}$ with $i \geqslant 2$. Then $F^i(x_i) \in K(f)$, but $F^j(x_i) \notin K(f)$ if $j \in \{0, \dots, i-1\}$. Since $f^{-n}(K(f)) \subseteq K(f)$ for all $n \in \mathbb{N}$, it follows that $F^j(x_i) \notin M$ if $j \in \{0, \dots, i-1\}$. Moreover since Lemma 5.3 (iii) implies $F([a_1, b_1]) \subseteq M$, we obtain $F^j(x_i) \notin [a_1, b_1]$ if $0 \leqslant j < i-1$. If $x_i \in [a_0, a_1)$, using the same inductive argument in (ii), we have $|b_0 - a_0| \geqslant \left|F^{i-1}([a_0, x_i])\right| > \theta^{i-1}|a_0 - x_i|$. Similarly, if $x_i \in (b_1, b_0]$, we also have $|b_0 - a_0| \geqslant \left|F^{i-1}([x_i, b_0])\right| > \theta^{i-1}|b_0 - x_i|$. Therefore, $\lim_{i\to+\infty} \min\{|a_0 - x_i|, |b_0 - x_i|\} = 0$, and hence the only possible limit points of $\{x_i\}_{i\in\mathbb{N}_0}$ are a_0 and b_0. This shows that K_0 is compact.

(iv) Fix $G \in W$. Since $G(\{a_0, b_0\}) \subseteq \{a_0, b_0\}$ and $G(M) \subseteq M$, it follows from (ii) that every G-invariant measure is a convex combination of a G-invariant measure supported on $\{a_0, b_0\}$ and a G-invariant measure supported on M. More precisely, for every $\mu \in \mathcal{M}(M_0, G)$, there exist $t_{\mu,1}, t_{\mu,2} \in [0,1]$ with $t_{\mu,1} + t_{\mu,2} = 1$ and $\mu_M, \mu_e \in \mathcal{M}(M_0, G)$ with $\operatorname{supp}\mu_M \subseteq M$ and $\operatorname{supp}\mu_e \subseteq \{a_0, b_0\}$ such that $\mu = t_{\mu,1}\mu_M + t_{\mu,2}\mu_e$. If $\phi \in C(M_0)$ satisfies $\phi(a_0) < Q(G, \phi)$ and $\phi(b_0) < Q(G, \phi)$, and if $\mu \in \mathcal{M}_{\max}(G, \phi)$, then μ_e can not be ϕ-maximizing. Hence $t_{\mu,2} = 0$, and therefore $\mu = \mu_M$ is supported on M. $\square$

After the above preparatory results, we are now ready to prove Theorem D.

Proof of Theorem D. If M is the circle, then Theorem D follows by combining Proposition 5.1, Lemma 5.2, [KSS07, Theorem 2], and the fact that hyperbolicity is an open property in $C^r(M, M)$ (see [dMvS93, Theorem 2.4, Chapter 3]), by an argument analogous to the proof of Theorem C.

We now assume that M is a compact interval and write $M = [a, b]$. Fix $k \in C^r(M, M)$, $\phi \in \mathcal{P}$, and $\varepsilon > 0$. Note that $\{l \in C^r(M, M) : \{l(a), l(b)\} \subseteq (a, b),\ l'(a) \neq 0 \text{ and } l'(b) \neq 0\}$ is open and dense in $C^r(M, M)$. Moreover, by [KSS07, Theorem 2], hyperbolic maps are dense in $C^r(M, M)$. Therefore, there exists a hyperbolic map $f \in C^r(M, M)$ such that $\{f(a), f(b)\} \subseteq (a, b)$, $f'(a) \neq 0$, $f'(b) \neq 0$, and

$$\|f - k\|_{C^r} < \varepsilon. \tag{5.3}$$

By Lemma 5.3 and Lemma 5.5 (iii), f admits a hyperbolic extension $F \in C_0^r([a_0, b_0], [a_0, b_0])$ with $a_0 < a < b < b_0$. Set $M_0 := [a_0, b_0]$, and define $\mathcal{P}_0 := C^{0,\alpha}(M_0, \mathbb{R})$ if $\mathcal{P} = C^{0,\alpha}(M, \mathbb{R})$, and

$\mathcal{P}_0 \coloneqq C^1(M_0, \mathbb{R})$ if $\mathcal{P} = C^1(M, \mathbb{R})$. If $\mathcal{P} = C^{0,\alpha}(M)$ for some $\alpha \in (0, 1]$, then for simplicity we write $\|\cdot\|_{\mathcal{P}} \coloneqq \|\cdot\|_{\alpha,M}$ and $\|\cdot\|_{\mathcal{P}_0} \coloneqq \|\cdot\|_{\alpha,M_0}$. Similarly, if $\mathcal{P} = C^1(M)$, then for simplicity we write $\|\cdot\|_{\mathcal{P}} \coloneqq \|\cdot\|_{C^1,M}$ and $\|\cdot\|_{\mathcal{P}_0} \coloneqq \|\cdot\|_{C^1,M_0}$.

Let $\Phi \in \mathcal{P}_0$ satisfy $\Phi|_M = \phi$, and

$$\Phi(a_0) < Q(f, \phi) - 1 \quad \text{and} \quad \Phi(b_0) < Q(f, \phi) - 1. \tag{5.4}$$

Since $F|_M = f$ and $F(M) = f(M) \subseteq M$, Lemma 5.5 (ii) guarantees that every F-invariant measure is supported on $M \cup \{a_0, b_0\}$. We claim that $Q(F, \Phi) = Q(f, \phi)$. Indeed, any (f, ϕ)-maximizing measure is F-invariant with Φ-average equal to $Q(f, \phi)$ (since $F|_M = f$ and $\Phi|_M = \phi$), so $Q(F, \Phi) \geqslant Q(f, \phi)$. Conversely, for each F-invariant measure μ, writing $\mu = t\mu_M + (1 - t)\mu_e$ where μ_M is supported on M, μ_e is supported on $\{a_0, b_0\}$, and $t \in [0, 1]$, we get from (5.4) that

$$\int \Phi \,\mathrm{d}\mu = t \int \phi \,\mathrm{d}\mu_M + (1 - t) \int \Phi \,\mathrm{d}\mu_e \leqslant t \cdot Q(f, \phi) + (1 - t) \cdot \max\{\Phi(a_0), \Phi(b_0)\} \leqslant Q(f, \phi).$$

Consequently $Q(F, \Phi) \leqslant Q(f, \phi)$, so indeed $Q(F, \Phi) = Q(f, \phi)$.

By Lemma 5.2, there exists $\Psi \in \mathrm{Lock}(F, \mathcal{P}_0)$ satisfying $\|\Phi - \Psi\|_{\mathcal{P}_0} < \min\{\varepsilon, 1/4\}$. Consequently,

$$\begin{aligned} \Psi(a_0) < \Phi(a_0) + 1/4 < Q(F, \Phi) - 3/4 < Q(F, \Psi) - 1/2, \\ \Psi(b_0) < \Phi(b_0) + 1/4 < Q(F, \Phi) - 3/4 < Q(F, \Psi) - 1/2. \end{aligned} \tag{5.5}$$

We may assume without loss of generality that Ψ is not a constant. Indeed, since $\mathrm{Lock}(F, \mathcal{P}_0)$ is open, we may replace Ψ by a sufficiently small perturbation satisfying the same properties. By Proposition 5.1, there exists a neighbourhood $U_1 \subseteq C^r_0(M_0, M_0) \times \mathcal{P}_0$ of (F, Ψ) such that $U_1 \subseteq \mathrm{Lock}(C^r_0(M_0, M_0), \mathcal{P}_0)$.

By Proposition 5.1, $HC^r_0(M_0, M_0)$ is an open subset of $C^r_0(M_0, M_0)$ containing F. Because F satisfies conditions (IS) and (ML) in Definition 2.1, and U_1 guarantees that $\Psi \in \mathrm{Lock}(G, \mathcal{P}_0)$ for all G in a neighbourhood of F, Remark 2.6 (ii) ensures that the map $Q(\cdot, \Psi)$ is continuous at F (noting that if $\mathcal{P}_0 = C^1(M_0, \mathbb{R})$ then we use the continuous embedding $C^1 \subseteq C^{0,1}$). Moreover, for each $G \in C^r_0(M_0, M_0)$, the map $Q(G, \cdot) \colon \mathcal{P}_0 \to \mathbb{R}$ is 1-Lipschitz with respect to the $\|\cdot\|_\infty$ norm. More precisely, for all $G \in C^r_0(M_0, M_0)$, ξ_1, $\xi_2 \in \mathcal{P}_0$, we have $|Q(G, \xi_1) - Q(G, \xi_2)| \leqslant \|\xi_1 - \xi_2\|_\infty \leqslant \|\xi_1 - \xi_2\|_{\mathcal{P}_0}$. Hence there exists a neighbourhood $U_2 \subseteq C^r_0(M_0, M_0) \times \mathcal{P}_0$ of (F, Ψ) such that $|Q(G, \Psi_1) - Q(F, \Psi)| < 1/4$ for every pair $(G, \Psi_1) \in U_2$.

Let $W \subseteq C^r_0(M_0, M_0)$ be the neighbourhood of F given by Lemma 5.5 (ii), and define $U_3 \coloneqq W \times B_{\mathcal{P}_0}(\Psi, 1/4)$. Set

$$U_0 \coloneqq U_1 \cap U_2 \cap U_3,$$

so that U_0 is a neighbourhood of (F, Ψ). Since $U_0 \subseteq U_1$, for each pair $(G, \Psi_1) \in U_0$, the (G, Ψ_1)-maximizing measure is unique and periodic. Moreover, by (5.5) we obtain

$$\begin{aligned} \Psi_1(a_0) < \Psi(a_0) + 1/4 < Q(F, \Psi) - 1/4 < Q(G, \Psi_1), \\ \Psi_1(b_0) < \Psi(b_0) + 1/4 < Q(F, \Psi) - 1/4 < Q(G, \Psi_1). \end{aligned} \tag{5.6}$$

Let $\kappa > 0$ be the constant in Lemma 5.5 (i) applied to F and M_0. Choose $\delta \in (0, \kappa)$ such that

$$B_{C^r_0(M_0,M_0)}(F, \delta) \times B_{\mathcal{P}_0}(\Psi, \delta) \subseteq U_0. \tag{5.7}$$

Set $\psi \coloneqq \Psi|_M$. Then

$$\|\phi - \psi\|_{\mathcal{P}} \leqslant \|\Phi - \Psi\|_{\mathcal{P}_0} < \varepsilon. \tag{5.8}$$

We now show that ψ belongs to the interior of $\mathrm{Lock}(C^r(M, M), \mathcal{P})$. Applying Lemma 5.4 to $M \subseteq M_0$, we obtain constants $D_1 \geqslant 1$ and $D_r \geqslant 1$. Denote

$$V \coloneqq B_{C^r(M,M)}(f, \delta/D_r) \times B_{\mathcal{P}}(\psi, \delta/D_1),$$

and choose an arbitrary pair $(g, \xi) \in V$.

By Lemma 5.4, the map $g-f$ admits an extension $H \in C^r(M_0,\mathbb{R})$ satisfying $H(a_0)=H(b_0)=0$ and

$$\|H\|_{C^r,M_0} \leqslant D_r\|g-f\|_{C^r,M} \leqslant \delta < \kappa. \tag{5.9}$$

Define $G \coloneqq F+H$. Then $G|_M = g$, and by (5.9) together with Lemma 5.5 (i), we have $G \in C^r_0(M_0,M_0)$. On the other hand, if $\mathcal{P} = C^{0,\alpha}(M,\mathbb{R})$, then by [Wea18, Theorem 1.33] there exists an extension $\zeta_1 \in C^{0,\alpha}(M_0,\mathbb{R})$ of $\xi-\psi$ such that $\|\zeta_1\|_{\alpha,M_0} \leqslant \|\xi-\psi\|_{\alpha,M}$. If $\mathcal{P}=C^1(M,\mathbb{R})$, then by Lemma 5.4 there exists an extension $\zeta_2 \in C^1(M_0,\mathbb{R})$ of $\xi-\psi$ such that $\|\zeta_2\|_{C^1,M_0} \leqslant D_1\|\xi-\psi\|_{C^1,M}$.

Since $D_1 \geqslant 1$, in either case there exists an extension $\zeta \in \mathcal{P}_0$ of $\xi-\psi$ satisfying

$$\|\zeta\|_{\mathcal{P}_0} \leqslant D_1\|\xi-\psi\|_{\mathcal{P}} \leqslant \delta. \tag{5.10}$$

Define $\Xi \coloneqq \Psi+\zeta$. Then $\Xi|_M = \xi$. Moreover, by (5.9) and (5.10),

$$(G,\Xi) \in B_{C^r_0(M_0,M_0)}(F,\delta) \times B_{\mathcal{P}_0}(\Psi,\delta).$$

Hence by (5.7), $(G,\Xi) \in U_0 \subseteq U_1$. Because $U_1 \subseteq \mathrm{Lock}(C^r_0(M_0,M_0),\mathcal{P}_0)$, the pair (G,Ξ) has the locking property: its maximizing measure is unique and periodic, and $\mathcal{M}_{\max}(G,\Xi_1) = \mathcal{M}_{\max}(G,\Xi)$ for every $\Xi_1 \in \mathcal{P}_0$ sufficiently close to Ξ.

Now $G|_M = g \in C^r(M,M)$, so $G(M) \subseteq M$. Since $(G,\Xi) \in U_0 \subseteq U_3$, we also know that $G \in W$. Moreover, the bounds in (5.6) establish that $\Xi(a_0) < Q(G,\Xi)$ and $\Xi(b_0) < Q(G,\Xi)$. Combining these, Lemma 5.5 (iv) gives that the unique (G,Ξ)-maximizing measure, μ^* say, has support contained in M. Since $G|_M = g$ and $\Xi|_M = \xi$, it follows that μ^* is precisely the unique (g,ξ)-maximizing measure.

It remains to verify that (g,ξ) has the locking property (rather than merely the PO property). Given any $\xi_1 \in \mathcal{P}$ sufficiently close to ξ, by Lemma 5.4 (if $\mathcal{P}=C^1(M,\mathbb{R})$) or by [Wea18, Theorem 1.33] (if $\mathcal{P}=C^{0,\alpha}(M,\mathbb{R})$), there exists an extension $\zeta_1 \in \mathcal{P}_0$ of $\xi_1-\xi$ satisfying $\|\zeta_1\|_{\mathcal{P}_0} \leqslant D_1\|\xi_1-\xi\|_{\mathcal{P}}$. Setting $\Xi_1 \coloneqq \Xi+\zeta_1$, we have $\Xi_1|_M = \xi_1$ and $\|\Xi_1-\Xi\|_{\mathcal{P}_0} \leqslant D_1\|\xi_1-\xi\|_{\mathcal{P}}$, so by choosing ξ_1 sufficiently close to ξ, we can make the extension Ξ_1 arbitrarily close to Ξ. The locking property of (G,Ξ) then yields $\mathcal{M}_{\max}(G,\Xi_1) = \mathcal{M}_{\max}(G,\Xi) = \{\mu^*\}$. Since μ^* is supported on M and $G|_M = g$, the measure μ^* is g-invariant with $\int \xi_1 \,\mathrm{d}\mu^* = \int \Xi_1 \,\mathrm{d}\mu^* = Q(G,\Xi_1)$. Conversely, any (g,ξ_1)-maximizing measure ν is also (G,Ξ_1)-maximizing (since ν is G-invariant and $\int \Xi_1 \,\mathrm{d}\nu = \int \xi_1 \,\mathrm{d}\nu = Q(g,\xi_1) = Q(G,\Xi_1)$), so $\nu = \mu^*$ by uniqueness. Hence $\mathcal{M}_{\max}(g,\xi_1) = \{\mu^*\} = \mathcal{M}_{\max}(g,\xi)$, and therefore $(g,\xi) \in \mathrm{Lock}(C^r(M,M),\mathcal{P})$.

Since $(g,\xi) \in V$ was arbitrary, we conclude that (f,ψ) belongs to the interior of $\mathrm{Lock}(C^r(M,M),\mathcal{P})$.

Finally, by (5.3), (5.8), the arbitrariness of $\varepsilon > 0$, $k \in C^r(M,M)$, and $\phi \in \mathcal{P}$, we conclude that the interior of $\mathrm{Lock}(C^r(M,M),\mathcal{P})$ is an open dense subset of $C^r(M,M)\times\mathcal{P}$. This completes the proof of Theorem D. □

We conclude this section with the proof of Theorem E.

Proof of Theorem E. Since $g_a(x) = ax(1-x)$ is linear in a, the topology on $\mathcal{F}$ coincides with the subspace topology inherited from $C^1([0,1],[0,1])$. Since $g_a(0)=g_a(1)=0$ for each $a \in [0,4]$, every map in $\mathcal{F}$ preserves the boundary of $[0,1]$, so $\mathcal{F} \subseteq C^1_0([0,1],[0,1])$.

Let $\mathcal{HF} \subseteq \mathcal{F}$ denote the subset of hyperbolic maps. By Graczyk–Świątek [GS97] and Lyubich [Ly97, p. 4], $\mathcal{HF}$ is open and dense in $\mathcal{F}$. Since $\mathcal{HF}$ is open and dense in $\mathcal{F}$, the product $\mathcal{HF}\times\mathcal{P}$ is open and dense in $\mathcal{F}\times\mathcal{P}$. It therefore suffices to show that the joint locking set $\mathrm{Lock}(\mathcal{HF},\mathcal{P})$ contains an open dense subset of $\mathcal{HF}\times\mathcal{P}$.

To show that maps in $\mathcal{HF}$ are $\mathcal{HF}$-stably hyperbolic, fix $f \in \mathcal{HF}$. Since $f \in HC^1_0([0,1],[0,1])$ and $\mathcal{HF} \subseteq C^1_0([0,1],[0,1])$, Proposition 5.1 guarantees that f is $C^1_0([0,1],[0,1])$-stably hyperbolic in the sense of Remark 2.6 (i). Restricting the neighbourhood of $C^1_0([0,1],[0,1])$ furnished by Definition 2.1 to the smaller family $\mathcal{HF} \subseteq C^1_0([0,1],[0,1])$, it follows that f is $\mathcal{HF}$-stably hyperbolic.

Now note that Individual TPO holds for maps in $\mathcal{HF}$: by Lemma 5.2, for every $f \in \mathcal{HF}$, the locking set $\mathrm{Lock}(f,\mathcal{P})$ is an open dense subset of $\mathcal{P}$.

We now claim that the set

$$\mathcal{L}_0 \coloneqq \{(g,\phi) \in \operatorname{Lock}(\mathcal{HF},\, \mathcal{P}) : \phi \text{ is nonconstant}\}$$

is an open dense subset of $\mathcal{HF} \times \mathcal{P}$.

To check openness, suppose $(g,\phi) \in \mathcal{L}_0$. Then g is $\mathcal{HF}$-stably hyperbolic by the above, and ϕ is nonconstant, so Theorem 2.5 implies that (g,ϕ) lies in the interior of $\operatorname{Lock}(\mathcal{HF},\mathcal{P})$. Clearly, being nonconstant is an open property in $\mathcal{P}$, so (g,ϕ) lies in the interior of $\mathcal{L}_0$. Since $(g,\phi) \in \mathcal{L}_0$ was arbitrary, it follows that $\mathcal{L}_0$ is open in $\mathcal{HF} \times \mathcal{P}$.

To check density of $\mathcal{L}_0$ in $\mathcal{HF}\times\mathcal{P}$, let $\mathcal{W}$ be any nonempty open set in $\mathcal{HF}\times\mathcal{P}$. Then $\mathcal{W}$ contains a basic open set of the form $\mathcal{V} \times \mathcal{B}$, where $\mathcal{V}$ is nonempty and open in $\mathcal{HF}$, and $\mathcal{B}$ is nonempty and open in $\mathcal{P}$. Choose any $f \in \mathcal{V}$. We know that $\operatorname{Lock}(f,\mathcal{P})$ is dense in $\mathcal{P}$, by the above, so there exists a nonconstant potential $\phi \in \mathcal{B} \cap \operatorname{Lock}(f,\mathcal{P})$. It follows that $(f,\phi) \in \mathcal{L}_0 \cap \mathcal{W}$, so indeed $\mathcal{L}_0$ is dense in $\mathcal{HF} \times \mathcal{P}$.

Since $\mathcal{L}_0$ is open and dense in $\mathcal{HF}\times\mathcal{P}$, and $\mathcal{HF}\times\mathcal{P}$ is open and dense in $\mathcal{F}\times\mathcal{P}$, the set $\mathcal{L}_0$ is in particular dense in $\mathcal{F}\times\mathcal{P}$. Moreover, every element of $\mathcal{L}_0$ lies in the interior of $\operatorname{Lock}(\mathcal{F},\mathcal{P})$ (since $\mathcal{HF}$ is open in $\mathcal{F}$, any $\mathcal{HF}$-locking neighbourhood is also a $\mathcal{F}$-locking neighbourhood), therefore $\operatorname{Lock}(\mathcal{F},\mathcal{P})$ contains an open dense subset of $\mathcal{F}\times\mathcal{P}$; in other words $\mathcal{F}\times\mathcal{P}$ has the Joint TPO property, as required. □

Appendix A. Proof of some technical lemmas

First, we give a complete proof of Lemma 3.3.

Let $f\colon M \to M$ be an Axiom A diffeomorphism with the splitting $T_xM = E^s(x) \oplus E^u(x)$. A Riemannian metric on M is said to be *adapted to* f if there exists a constant $\tau_f \in (0,1)$ such that for the induced norm $|\cdot|$, and every $x \in \Omega(f)$,

$$|\mathrm{D}f(x)(v)| \leqslant \tau_f|v| \text{ for all } v \in E^s(x) \quad \text{ and } \quad \left|\mathrm{D}f^{-1}(x)(w)\right| \leqslant \tau_f|w| \text{ for all } w \in E^u(x).$$

By [Wen16, Theorem 4.4], there exists a smooth Riemannian metric adapted to f. Furthermore, we denote by $\|\cdot\|_f$ the *box-adjusted norm* of $|\cdot|$, defined as

$$\|v\|_f \coloneqq \max\{|v_s|,\, |v_u|\} \quad \text{ for } x \in \Omega(f) \text{ and } v \in T_xM, \tag{A.1}$$

where $v = v_s + v_u$ is the unique splitting with $v_s \in E^s(x)$ and $v_u \in E^u(x)$.

Proof of Lemma 3.3. Fix $r \in \mathbb{N}$ and $f \in \mathcal{A}^r(M)$. We consider a Riemannian metric $|\cdot|_a$ that is adapted to f, and use the distance d_a induced by $|\cdot|_a$. Since M is compact, both $|\cdot|$ and $|\cdot|_a$ are equivalent, as are d and d_a, so it is sufficient to prove condition (RHE) in Definition 2.1 for d_a. Throughout this proof, we always consider exponential maps and Lipschitz constants with respect to $|\cdot|_a$ and d_a.

For every $x \in M$, let $\exp_x$ be the exponential map at x, and we write $T_xM(\delta) \coloneqq \{v \in T_xM : |v|_a \leqslant \delta\}$ for every $\delta > 0$.

Recall the following well-known result in Riemannian geometry (see e.g. [Le18, Chapter 10]): there exists a constant $\rho > 0$ (the injectivity radius) such that for every $x \in M$, $\exp_x\colon T_xM(\rho) \to \overline{B(x,\rho)}$ is a homeomorphism and $d_a(x, \exp_x(v)) = |v|_a$ for all $v \in T_xM(\rho)$.

For every Axiom A diffeomorphism $g\colon M \to M$, we consider the local map on the tangent spaces

$$F_g(x,v) = \left(\exp_{g(x)}^{-1} \circ g \circ \exp_x\right)(v) \quad \text{ if } x \in M \text{ and } v \in T_xM(\rho/\operatorname{LIP}(g)),$$

where $\exp_{g(x)}^{-1}$ denotes the inverse of $\exp_{g(x)}|_{T_{g(x)}M(\rho)}$. Since g is a diffeomorphism, $g(M) = M$, so $\operatorname{diam}(M) = \operatorname{diam}(g(M)) \leqslant \operatorname{LIP}(g)\operatorname{diam}(M)$, therefore $\operatorname{LIP}(g) \geqslant 1$.

By the definition of F_g, if x, $y \in M$ and $n \in \mathbb{N}$ satisfy $d_a\big(g^i(x), g^i(y)\big) < \rho/\operatorname{LIP}(g)$ for all $0 \leqslant i \leqslant n-1$, then letting $v \in T_xM(\rho)$ satisfy $\exp_x(v) = y$ gives $\exp_{g^i(x)}\big(F_g^i(x,v)\big) = g^i(y)$ and

$d_a\big(g^i(x), g^i(y)\big) < \rho$ for all $0 \leqslant i \leqslant n$. So if $0 \leqslant i \leqslant n$ then $F_g^i(x,v)$ is well defined and

$$d_a\big(g^i(x), g^i(y)\big) = \big|F_g^i(x,v)\big|_a. \tag{A.2}$$

In the following, we construct a neighbourhood U of f. First, define

$$\eta \coloneqq \rho/(2\,\mathrm{LIP}(f)). \tag{A.3}$$

There exists a C^1 neighbourhood (hence also a C^r neighbourhood) U_0 of f such that if $g \in U_0$ then

$$\mathrm{LIP}(g) \leqslant 2\,\mathrm{LIP}(f). \tag{A.4}$$

For each $g \in U_0$, the map F_g is well defined on $TM(\eta) \coloneqq \bigcup_{x\in M} T_xM(\eta)$ by (A.3) and (A.4). For each $x \in M$ and $v \in T_xM(\eta)$, define the nonlinear remainder

$$\phi_g(x,v) \coloneqq \exp_{g(x)}^{-1}\big(g(\exp_x(v))\big) - \mathrm{D}g(x)(v) \in T_{g(x)}M,$$

so that ϕ_g is also well defined on $TM(\eta)$ for $g \in U_0$. Note that $\phi_g(x,0) = 0 \in T_{g(x)}M$ for every $x \in M$.

Next, applying [Wen16, Lemma 4.8] to f and $|\cdot|_a$, there exist constants $\varepsilon > 0$, $C > 1$, and a C^1 neighbourhood (hence also a C^r neighbourhood) U_1 of f such that if $g \in U_1$ and $\Omega(g) \subseteq B(\Omega(f), \varepsilon)$ then $|\cdot|_a$ is also adapted to g, with

$$\tau_g < \tau_f + 3^{-1}(1-\tau_f) < 1, \tag{A.5}$$

and the norm $\|\cdot\|_g$ defined by (A.1) on $\Omega(g)$ satisfies

$$C^{-1}\|v\|_g \leqslant |v|_a \leqslant C\|v\|_g \quad \text{for } x \in \Omega(g) \text{ and } v \in T_xM(\eta), \tag{A.6}$$

where the constant C is uniform in $g \in U_1$. By [Wen16, Lemma 4.11] applied to f, there exists a constant $\delta_0 > 0$ and a C^1 neighbourhood (hence also a C^r neighbourhood)[1] $U_2 \subseteq U_0$ of f such that if $g \in U_2$ then

$$\mathrm{LIP}^{|\cdot|_a}_{v,\delta_0}(\phi_g) < 3^{-1}C^{-2}(1-\tau_f), \tag{A.7}$$

where $\mathrm{LIP}^{|\cdot|_a}_{v,\delta_0}(\phi_g) \coloneqq \sup_{x\in M}\big(\mathrm{LIP}_{|\cdot|_a}\, \phi_g(x,\cdot)|_{T_xM(\delta_0)}\big)$. Finally, since f has C^r ε-Ω-stability ([Wen16, Theorem 5.8]), there exists a C^r neighbourhood U_3 of f such that if $g \in U_3$ then $\Omega(g) \subseteq B(\Omega(f), \varepsilon)$. Now define $U \coloneqq U_0 \cap U_1 \cap U_2 \cap U_3$, so that (A.4), (A.5), (A.6), (A.7) hold for all $g \in U$.

Now define constants

$$K_0 \coloneqq C^2, \tag{A.8}$$

$$\delta \coloneqq \min\{\eta,\, \delta_0\}, \tag{A.9}$$

$$\lambda \coloneqq \min\big\{3(2+\tau_f)^{-1},\, 3(1+2\tau_f)^{-1} - 3^{-1}(1-\tau_f)\big\}, \tag{A.10}$$

noting that $\lambda > 1$ since $0 < \tau_f < 1$.

Next we shall verify that the inequality (2.1) from condition (RHE) in Definition 2.1 holds for all $g \in U$. Fix $g \in U$, $n \in \mathbb{N}$, $x \in \Omega(g)$, and $y \in M$ with $\max_{0\leqslant i\leqslant n} d_a(g^i(x), g^i(y)) \leqslant \delta$. Now $d_a(x,y) \leqslant \delta \leqslant \eta \leqslant \rho$, so $v \coloneqq \exp_x^{-1}(y) \in T_xM(\rho)$ is well defined. We verify by induction that $F_g^i(x,v)$ is well defined for all $0 \leqslant i \leqslant n$, and that (A.2) holds at each step. At $i = 0$ both are immediate from $\exp_x(v) = y$. Assuming (A.2) holds at step i, we have

$$d_a(g^i(x), g^i(y)) = \big|F_g^i(x,v)\big|_a \leqslant \delta \leqslant \eta = \rho/(2\,\mathrm{LIP}(f)) \leqslant \rho/\mathrm{LIP}(g),$$

using (A.9), (A.3), and (A.4), so $F_g^{i+1}(x,v)$ is well defined and (A.2) holds at step $i+1$. Since $\exp_x\colon T_xM(\delta) \to B_{(M,d_a)}(x,\delta)$ is a homeomorphism and (A.2) identifies $d_a(g^i(x), g^i(y))$ with $\big|F_g^i(x,v)\big|_a$, it now suffices to prove that

$$\big|F_g^i(x,v)\big|_a \leqslant K_0\lambda^{-\min\{i,\,n-i\}}\big(|v|_a + \big|F_g^n(x,v)\big|_a\big). \tag{A.11}$$

[1] We choose U_2 to be contained in U_0, in order that ϕ_g is well-defined for all $g \in U_2$.

By (A.6), (A.9), and (A.7), we have

$$\mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g) \leqslant C^2 \,\mathrm{LIP}_{v,\delta}^{|\cdot|_a}(\phi_g) \leqslant C^2 \,\mathrm{LIP}_{v,\delta_0}^{|\cdot|_a}(\phi_g) < 3^{-1}(1-\tau_f),$$

where $\mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g) := \sup_{z\in M}\big(\mathrm{LIP}_{\|\cdot\|_g}\,\phi_g(z,\cdot)|_{T_zM(\delta)}\big)$. Combining this with (A.5) and (A.10) gives

$$\begin{aligned} \tau_g + \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g) &< \tau_f + 3^{-1}(1-\tau_f) + 3^{-1}(1-\tau_f) < \lambda^{-1} \quad \text{and} \\ \tau_g^{-1} - \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g) &> 3(1+2\tau_f)^{-1} - 3^{-1}(1-\tau_f) > \lambda. \end{aligned} \tag{A.12}$$

For each $z \in \Omega(g)$, let $T_zM = E_g^s(z) \oplus E_g^u(z)$ be the hyperbolic splitting of g, and for every $w \in T_zM$, write $w = w_s + w_u$ for the unique decomposition with $w_s \in E_g^s(z)$ and $w_u \in E_g^u(z)$. Since $\phi_g(z,0) = 0$, the definition of $\mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g)$ and (A.1) give, for each $z \in \Omega(g)$ and $w \in T_zM(\delta)$,

$$|(\phi_g(z,w))_\sigma|_a \leqslant \|\phi_g(z,w)\|_g \leqslant \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g)\,\|w\|_g, \quad \text{for each } \sigma \in \{s,u\}. \tag{A.13}$$

Then by (A.1), the fact that $|\cdot|_a$ is adapted to g, (A.13), and (A.12), for each $z \in \Omega(g)$ and each $w \in T_zM(\delta)$, we obtain

$$\begin{aligned} |(F_g(z,w))_s|_a &\leqslant |\mathrm{D}g(z)(w_s)|_a + |(\phi_g(z,w))_s|_a \leqslant \tau_g|w_s|_a + |(\phi_g(z,w))_s|_a \\ &\leqslant \tau_g\|w\|_g + \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g)\|w\|_g < \lambda^{-1}\|w\|_g. \end{aligned} \tag{A.14}$$

Similarly, $\mathrm{D}g(z)(w_u) \in E_g^u(g(z))$, so $|\mathrm{D}g(z)(w_u)|_a \geqslant \tau_g^{-1}|w_u|_a$. Since $(F_g(z,w))_u = \mathrm{D}g(z)(w_u) + (\phi_g(z,w))_u$, it follows from (A.13) and (A.12) that

$$|(F_g(z,w))_u|_a \geqslant |\mathrm{D}g(z)(w_u)|_a - |(\phi_g(z,w))_u|_a \geqslant \tau_g^{-1}|w_u|_a - \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g)\,\|w\|_g. \tag{A.15}$$

Next, we consider separately the following two cases.

Case 1. Assume that $\big|\big(F_g^i(x,v)\big)_u\big|_a \leqslant \big|\big(F_g^i(x,v)\big)_s\big|_a$ for all $0 \leqslant i \leqslant n$. In this case, we conclude from (A.14) that for each $1 \leqslant i \leqslant n$,

$$\begin{aligned} \big\|F_g^i(x,v)\big\|_g = \big|\big(F_g^i(x,v)\big)_s\big|_a &< \lambda^{-1}\big|\big(F_g^{i-1}(x,v)\big)_s\big|_a < \cdots < \lambda^{-i}\|v\|_g \\ &\leqslant \lambda^{-\min\{i,\,n-i\}}\big(\|v\|_g + \big\|F_g^n(x,v)\big\|_g\big), \end{aligned} \tag{A.16}$$

and the case $i = 0$ holds trivially.

Case 2. Assume that $\big|\big(F_g^k(x,v)\big)_u\big|_a > \big|\big(F_g^k(x,v)\big)_s\big|_a$ for some $0 \leqslant k \leqslant n$, where k is chosen to be the smallest number with this property. An argument analogous to Case 1 gives that if $i \notin \{k,\,\ldots,\,n\}$, then

$$\big\|F_g^i(x,v)\big\|_g < \lambda^{-i}\|v\|_g. \tag{A.17}$$

Since $\big|\big(F_g^k(x,v)\big)_u\big|_a = \big\|F_g^k(x,v)\big\|_g$, by (A.15), (A.12), and (A.14), we obtain that if $k \leqslant n-1$ then

$$\big|\big(F_g^{k+1}(x,v)\big)_u\big|_a > \big(\tau_g^{-1} - \mathrm{LIP}_{v,\delta}^{\|\cdot\|_g}(\phi_g)\big)\big\|F_g^k(x,v)\big\|_g > \lambda\big\|F_g^k(x,v)\big\|_g > \big|\big(F_g^{k+1}(x,v)\big)_s\big|_a. \tag{A.18}$$

Since $\big|(F_g^i(x,v))_u\big|_a > \big|(F_g^i(x,v))_s\big|_a$ for all $k \leqslant i \leqslant n$ (which follows by iterating (A.18) forward from k), the backward iteration of (A.18) from step n gives, for all $k \leqslant i \leqslant n-1$,

$$\big\|F_g^i(x,v)\big\|_g = \big|(F_g^i(x,v))_u\big|_a \leqslant \lambda^{i-n}\big\|F_g^n(x,v)\big\|_g. \tag{A.19}$$

Clearly, (A.19) is also true when $k = n$. In this case, we conclude from (A.17) and (A.19) that if $0 \leqslant i \leqslant n$ then

$$\big\|F_g^i(x,v)\big\|_g \leqslant \lambda^{-\min\{i,\,n-i\}}\big(\|v\|_g + \big\|F_g^n(x,v)\big\|_g\big). \tag{A.20}$$

Finally, combining (A.16), (A.20), (A.8), (A.6) gives (A.11). Together with (A.2) and the fact that $\exp_x\colon T_xM(\delta) \to B_{d_a}(x,\delta)$ is a homeomorphism, this establishes the required (2.1) for the distance d_a with constant K_0. Since M is compact, the original metric d and the adapted metric

d_a are bi-Lipschitz equivalent. Adjusting the constants δ and K_0 by the bi-Lipschitz equivalence constants immediately yields (2.1) for the metric d, thus completing the proof. □

Next we consider Lemma 4.3, which is essentially a one-sided version of Lemma 3.3: since $f|_{J(f)}$ is expanding (with no stable subbundle), only the expanding half of the hyperbolic splitting argument is required, and the proof proceeds by a straightforward adaptation of the argument above. We therefore give only a sketch, referring the reader to the proof of Lemma 3.3 for the full details of the analogous steps.

Proof of Lemma 4.3. We first show that the restriction $f|_{J(f)}\colon J(f) \to J(f)$ is an open map. Since $|\mathrm{D}f| > \lambda > 0$ on $J(f)$, there exists an open neighbourhood $V \subseteq \widehat{\mathbb{C}}$ of the Julia set on which the derivative is nonvanishing; hence, the restriction $f|_V : V \to \widehat{\mathbb{C}}$ is an open map. The complete invariance of the Julia set, specifically the fact that $f^{-1}(J(f)) = J(f)$ (see [Mi06, Lemma 4.3]), ensures that for each open set $U \subseteq V$, we have $f(U \cap J(f)) = f(U) \cap J(f)$. Because $f(U)$ is open in $\widehat{\mathbb{C}}$, its intersection with $J(f)$ is open in the subspace topology, thus $f|_{J(f)}$ is an open map.

It remains to prove the second assertion. Let exp be the exponential map induced by $|\cdot|$. Firstly, there exists $\eta_0 > 0$ such that $\exp_x\colon T_x\widehat{\mathbb{C}}(\eta_0) \to \overline{B(x,\eta_0)}$ is a homeomorphism and $d(x, \exp_x(v)) = |v|$ for every $x \in \widehat{\mathbb{C}}$ and every $v \in T_x\widehat{\mathbb{C}}(\eta_0)$. For $g \in \mathcal{HR}^m$, $x \in \widehat{\mathbb{C}}$, and $v \in T_x\widehat{\mathbb{C}}(\eta_0/\operatorname{LIP}(g))$, define $F_g(x,v) \coloneqq \bigl(\exp_{g(x)}^{-1} \circ g \circ \exp_x\bigr)(v)$, where $\exp_{g(x)}^{-1}$ denotes the inverse of $\exp_{g(x)}|_{T_{g(x)}\widehat{\mathbb{C}}(\eta_0)}$. Then define $\phi_g \coloneqq F_g - Tg$, where Tg is the tangent map of g.

Since $\mathrm{D}g(x)$ depends continuously on $g \in \mathcal{HR}^m$ and $x \in \widehat{\mathbb{C}}$, and the spaces $\widehat{\mathbb{C}}$ and $J(f)$ are compact, there exist constants $\delta > 0$, $\theta_0 > 1$, and a neighbourhood N_0 of f such that $\operatorname{LIP}(g) \leqslant 2\operatorname{LIP}(f)$ and $|\mathrm{D}g(x)| > \theta_0$ for every $g \in N_0$ and every $x \in B(J(f),\delta)$.

Then, using Lemma 4.2, there exists a neighbourhood N_1 of f such that if $g \in N_1$ then $J(g) \in B(J(f),\delta)$.

Next, given $\varepsilon \in (0, \theta_0 - 1)$, by [Wen16, Lemma 4.11], there exists a neighbourhood N_2 of f and $\delta_0 > 0$ such that if $g \in N_2$ then $\operatorname{LIP}^{|\cdot|}_{v,\delta_0}(\phi_g) < \varepsilon$ on $T\widehat{\mathbb{C}}(\delta_0)$.[2]

Finally, defining $\eta \coloneqq \min\{\eta_0/(2\operatorname{LIP}(f)),\, \delta_0\}$, $\theta \coloneqq \theta_0 - \varepsilon > 1$, and $N \coloneqq N_0 \cap N_1 \cap N_2$, if $g \in N$ and distinct points x, $y \in J(g)$ with $d(x,y) < \eta$, then we have

$$\begin{aligned} d(g(x), g(y)) &= \bigl|\exp_{g(x)}^{-1}(g(y))\bigr| = \bigl|F_g\bigl(\exp_x^{-1}(y)\bigr)\bigr| \geqslant \bigl|Tg\bigl(\exp_x^{-1}(y)\bigr)\bigr| - \bigl|\phi_g\bigl(\exp_x^{-1}(y)\bigr)\bigr| \\ &\geqslant \theta\bigl|\exp_x^{-1}(y)\bigr| = \theta d(x,y), \end{aligned}$$

and the result follows. □

Next, we consider Lemmas 5.3 and 5.4, concerning extensions of C^r interval maps. Before proving these lemmas, we first describe the construction of such extensions.

Consider $r \in \mathbb{N}$, $a < b$, and $f \in C^r([a,b],\mathbb{R})$. To construct an extension $F \in C^r(\mathbb{R},\mathbb{R})$, it suffices to find two functions $g \in C^r((-\infty,a],\mathbb{R})$ and $h \in C^r([b,+\infty),\mathbb{R})$ such that for each integer $0 \leqslant i \leqslant r$,

$$g^{(i)}(a) = f^{(i)}(a) \text{ and } h^{(i)}(b) = f^{(i)}(b),$$

where $f^{(i)}$ denotes the i-th derivative of f; then defining F by $F = f$ on $[a,b]$, $F = g$ on $(-\infty,a)$, and $F = h$ on $(b,+\infty)$ yields the desired extension.

[2]The statement of [Wen16, Lemma 4.11] is given for diffeomorphisms, but an inspection of its proof shows that the argument carries over unchanged to local diffeomorphisms, and in particular to hyperbolic rational maps.

In what follows, we restrict our attention to functions $g \in C^r((-\infty, a], \mathbb{R})$ and $h \in C^r([b, +\infty), \mathbb{R})$ of the following form: for each $k \in \mathbb{R}$, define

$$g_k(x) = \sum_{i=0}^{r} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{(-1)^{r+1}k}{(r+1)!}(x-a)^{r+1}, \tag{A.21}$$

$$h_k(x) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!}(x-b)^i + \frac{k}{(r+1)!}(x-b)^{r+1}. \tag{A.22}$$

Clearly, for each $k \in \mathbb{R}$ and each $i \in \{0, 1, \ldots, r\}$, we have $g_k^{(i)}(a) = f^{(i)}(a)$ and $h_k^{(i)}(b) = f^{(i)}(b)$, so g_k and h_k satisfy the required matching conditions. Moreover, for every $x \in (-\infty, a]$, the function $g_k(x)$ is linear and monotone increasing with respect to k, and for every $x \in [b, +\infty)$, the function $h_k(x)$ is also linear and monotone increasing with respect to k.

Proof of Lemma 5.3. Clearly $\tau \in (0, \min\{f(a) - a, b - f(b)\})$.

Claim A. If $f'(b) > 0$, then there exist $l_+ \in \mathbb{R}$ and $b_1 > b$ such that $h'_{l_+} > 0$ on $[b, +\infty)$, $h'_{l_+} > 2$ on $(b_1, +\infty)$, and $h_{l_+}(b_1) = b - \tau$.

Proof of Claim A. Since $h'_0(b) = f'(b) > 0$ and h'_0 is continuous, there exists $\theta > 0$ such that $h'_0 > 0$ on $[b, b+\theta)$. Hence, for all $k \in [0, +\infty)$ and $x \in [b, b+\theta)$, it follows from (A.22) that

$$h'_k(x) = h'_0(x) + \frac{k}{r!}(x-b)^r > 0. \tag{A.23}$$

For each $w \in (0, \theta)$, define

$$k_w := \frac{r!}{w^r}\left(\|f\|_{C^r}\sum_{i=0}^{r-1}\frac{w^i}{i!} + 2\right) > 0. \tag{A.24}$$

If $x > b + w$, a direct estimation using (A.22) and (A.24) gives

$$\begin{aligned} h'_{k_w}(x) &\geqslant \frac{k_w}{r!}(x-b)^r - \|f\|_{C^r}\sum_{i=0}^{r-1}\frac{(x-b)^i}{i!} \geqslant \frac{2(x-b)^r}{w^r} + \|f\|_{C^r}\sum_{i=0}^{r-1}\left(\frac{(x-b)^r w^i}{i! w^r} - \frac{(x-b)^i}{i!}\right) \\ &> 2 + \|f\|_{C^r}\sum_{i=0}^{r-1}\frac{(x-b)^i}{i!}\left(\frac{(x-b)^{r-i}}{w^{r-i}} - 1\right) \geqslant 2. \end{aligned} \tag{A.25}$$

Next, since $\lim_{w\to 0} h_0(b+w) = f(b)$ and, by (A.24), $\lim_{w\to 0} k_w w^{r+1} = 0$, it follows from (A.22) that

$$\lim_{w\to 0} h_{k_w}(b+w) = \lim_{w\to 0}\left(h_0(b+w) + \frac{k_w}{(r+1)!}w^{r+1}\right) = f(b) < b - \tau. \tag{A.26}$$

Hence, there exists $w_0 \in (0, \theta)$ such that $h_{k_{w_0}}(b + w_0) < b - \tau$. By (A.25), there exists $b_1 > b + w_0$ such that $h_{k_{w_0}}(b_1) = b - \tau$. Define $l_+ := k_{w_0}$. Combining (A.23), (A.25), and the fact that $w_0 < \theta$, we have $h'_{l_+} > 0$ on $[b, +\infty)$. Combining (A.25) and the fact that $b_1 > b + w_0$ gives $h'_{l_+} > 2$ on $(b_1, +\infty)$, thus completing the proof of Claim A.

Arguing similarly, the following three claims can also be established.

Claim B. If $f'(b) < 0$, then there exist $l_+ \in \mathbb{R}$ and $b_1 > b$ such that $h'_{l_+} < 0$ on $[b, +\infty)$, $h'_{l_+} < -2$ on $(b_1, +\infty)$, and $h_{l_+}(b_1) = a + \tau$.

Claim C. If $f'(a) > 0$, then there exist $l_- \in \mathbb{R}$ and $a_1 < a$ such that $g'_{l_-} > 0$ on $(-\infty, a]$, $g'_{l_-} > 2$ on $(-\infty, a_1)$, and $g_{l_-}(a_1) = a + \tau$.

Claim D. If $f'(a) < 0$, then there exist $l_- \in \mathbb{R}$ and $a_1 < a$ such that $g'_{l_-} < 0$ on $(-\infty, a]$, $g'_{l_-} < -2$ on $(-\infty, a_1)$, and $g_{l_-}(a_1) = b - \tau$.

Since $f'(a) \neq 0 \neq f'(b)$, combining Claims A, B, C, and D gives the following claim:

Claim E. There exist $l_+ \in \mathbb{R}$, $l_- \in \mathbb{R}$, $a_1 < a$, and $b_1 > b$ such that $\left|h'_{l_+}\right| \neq 0$ on $[b, +\infty)$, $\left|g'_{l_-}\right| \neq 0$ on $(-\infty, a]$, $\left|h'_{l_+}\right| > 2$ on $(b_1, +\infty)$, $\left|g'_{l_-}\right| > 2$ on $(-\infty, a_1)$, and $h_{l_+}(b_1)$, $g_{l_-}(a_1) \in \{a+\tau,\, b-\tau\}$.

Let $l_+ \in \mathbb{R}$, $l_- \in \mathbb{R}$, $b_1 > b$, and $a_1 < a$ be as in Claim E. Suppose a_0 and b_0 have been chosen such that $a_0 < a_1$ and $b_0 > b_1$. Define a function $F\colon [a_0, b_0] \to \mathbb{R}$ by

$$F(x) := \begin{cases} g_{l_-}(x) & \text{if } x \in [a_0, a), \\ f(x) & \text{if } x \in M, \\ h_{l_+}(x) & \text{if } x \in (b, b_0]. \end{cases} \tag{A.27}$$

By construction, $F \in C^r([a_0, b_0], \mathbb{R})$ and $F|_M = f$. Moreover, since $g'_{l_-} \neq 0$ on $(-\infty, a]$, and $h'_{l_+} \neq 0$ on $[b, +\infty)$, the function F is such that $F'(x) \neq 0$ for all $x \in [a_0, a] \cup [b, b_0]$, i.e., the required property (ii) of the lemma is satisfied.

We now show that property (iii) is also satisfied. Since $\tau \in (0, \min\{f(a) - a,\, b - f(b)\})$, then $f(b) \in (a+\tau, b-\tau)$, and by property (ii), the function h_{l_+} is monotone. Since $h_{l_+}(b_1)$ is equal to either $a + \tau$ or $b - \tau$, by Claim E, it follows that

$$h_{l_+}((b_1, b_0]) \cap [a+\tau, b-\tau] = \emptyset \quad \text{ and } \quad h_{l_+}([b, b_1]) \subseteq [a+\tau, b-\tau].$$

Moreover, Claim E also gives that $\left|h'_{l_+}\right| > 2$ on $(b_1, b_0]$. Arguing similarly, we also see that

$$g_{l_-}([a_0, a_1)) \cap [a+\tau, b-\tau] = \emptyset, \quad g_{l_-}([a_1, a]) \subseteq [a+\tau, b-\tau],$$

and $|g'_{l_-}| > 2$ on $[a_0, a_1)$, so F does indeed satisfy property (iii).

Note that property (i) (i.e., $\{F(a_0),\, F(b_0)\} \subseteq \{a_0,\, b_0\}$ and $F|_M = f$) is satisfied if additionally

$$\{g_{l_-}(a_0),\, h_{l_+}(b_0)\} \subseteq \{a_0,\, b_0\}. \tag{A.28}$$

Moreover, property (iii) then implies that $F([a_1, b_1]) \subseteq M$, and F is monotone on $[a_0, a_1]$ and $[b_1, b_0]$. If $\max\{F(x) : x \in [a_0, b_0]\} > b_0$, then the maximum is attained at a_0 or b_0, contradicting the above assumption. Similarly, $\min\{F(x) : x \in [a_0, b_0]\} \geqslant a_0$. Hence, $F([a_0, b_0]) \subseteq [a_0, b_0]$, and thus $F \in C^r_0([a_0, b_0], [a_0, b_0])$. So it remains to show that $a_0 < a_1$ and $b_0 > b_1$ can be chosen such that (A.28) holds.

For this we distinguish four cases according to the possible signs of $f'(a)$ and $f'(b)$.

Case 1. Assume that $f'(a) > 0$ and $f'(b) > 0$. Using Claim E and the assumption $f'(b) > 0$, we have $h_{l_+}(b_1) < b_1$ and $h'_{l_+} > 2$ on $(b_1, +\infty)$. Hence $x \mapsto h_{l_+}(x) - x$ is strictly increasing with derivative greater than 1, and is negative at $x = b_1$. Hence, there exists $b_0 > b_1$ such that $h_{l_+}(b_0) = b_0$. Similarly, there exists $a_0 < a_1$ such that $g_{l_-}(a_0) = a_0$.

Case 2. Assume that $f'(a) < 0$ and $f'(b) > 0$. Using Claim E and the assumption that $f'(b) > 0$, the same argument as in Case 1 yields $b_0 > b_1$ such that $h_{l_+}(b_0) = b_0$. Moreover, using Claim E and the assumption that $f'(a) < 0$, the function g_{l_-} is strictly decreasing on $(-\infty, a]$ with $g'_{l_-} < -2$ on $(-\infty, a_1)$; in particular, $g_{l_-}(x) \to +\infty$ as $x \to -\infty$. Since $g_{l_-}(a_1) < b < b_0$, the intermediate value theorem yields $a_0 < a_1$ such that $g_{l_-}(a_0) = b_0$.

Case 3. Assume that $f'(a) > 0$ and $f'(b) < 0$. This case is analogous to Case 2 and is omitted.

Case 4. Assume that $f'(a) < 0$ and $f'(b) < 0$. Using Claim E and our assumptions $f'(a) < 0$ and $f'(b) < 0$, we have $g'_{l_-} < 0$ on $(-\infty, a]$ and $h'_{l_+} < 0$ on $[b, +\infty)$. Then we seek $a_0 < a_1$ and $b_0 > b_1$ such that

$$g_{l_-}(a_0) = b_0 \text{ and } h_{l_+}(b_0) = a_0.$$

Set $p := b_1 - a_1$, $a_2 := a_1 - p$, and $b_2 := b_1 + p$. By the derivative bounds, one checks that

$$g_{l_-}(a_2) > b_2 \text{ and } h_{l_+}(b_2) < a_2.$$

More precisely, since $g_{l_-}(a_1) > a$ and $g'_{l_-} < -2$ on $[a_2, a_1]$, we have $g_{l_-}(a_2) > 2p + a > p + b_1 = b_2$. Similarly, $h_{l_+}(b_2) < a_2$.

Define $H(x,y) := |g_{l_-}(x) - y| + |h_{l_+}(y) - x|$ on $[a_2, a_1] \times [b_1, b_2]$.

Let $\mathrm{D}_x^- H$ and $\mathrm{D}_x^+ H$ respectively denote the left and right partial derivatives of H with respect to the first variable, and let $\mathrm{D}_y^- H$ and $\mathrm{D}_y^+ H$ respectively denote the left and right partial derivatives of H with respect to the second variable. Let $a_* \in [a_2, a_1]$ and $b_* \in [b_1, b_2]$ be such that $H(a_*, b_*) = \min\{H(x,y) : x \in [a_2, a_1],\, y \in [b_1, b_2]\}$. If $a_* = a_1$, the minimality of $H(a_*, b_*)$ implies $\mathrm{D}_x^- H(a_*, b_*) \leqslant 0$. However, since $g_{l_-}(a_1) < b < b_*$, we have $\mathrm{D}_x^- H(a_*, b_*) \geqslant -g_{l_-}'(a_1) - 1 > 1 > 0$, which leads to a contradiction. If $a_* = a_2$, the minimality of $H(a_*, b_*)$ implies $\mathrm{D}_x^+ H(a_*, b_*) \geqslant 0$. However, since $g_{l_-}(a_2) > b_2 \geqslant b_*$, we have $\mathrm{D}_x^+ H(a_*, b_*) \leqslant g_{l_-}'(a_2) + 1 < -1 < 0$, which leads to a contradiction. Thus, $a_* \in (a_2, a_1)$. Similarly, we have $b_* \in (b_1, b_2)$. Now assume that $g_{l_-}(a_*) \neq b_*$, by the minimality of $H(a_*, b_*)$, we have $\mathrm{D}_x^+ H(a_*, b_*) \geqslant 0$ and $\mathrm{D}_x^- H(a_*, b_*) \leqslant 0$. However, if $g_{l_-}(a_*) < b_*$, then $\mathrm{D}_x^+ H(a_*, b_*) \geqslant -g_{l_-}'(a_*) - 1 > 1 > 0$ and $\mathrm{D}_x^- H(a_*, b_*) \geqslant -g_{l_-}'(a_*) - 1 > 1 > 0$, which leads to a contradiction. If $g_{l_-}(a_*) > b_*$, then $\mathrm{D}_x^+ H(a_*, b_*) \leqslant g_{l_-}'(a_*) + 1 < -1 < 0$ and $\mathrm{D}_x^- H(a_*, b_*) \leqslant g_{l_-}'(a_*) + 1 < -1 < 0$, which leads to a contradiction. Thus $g_{l_-}(a_*) = b_*$. Similarly, $h_{l_+}(b_*) = a_*$. This completes the proof. □

Proof of Lemma 5.4. We first control the left extension. For $k \in \mathbb{R}$, recall the candidate extension g_k on $[a_0, a]$ defined (cf. (A.21)) by

$$g_k(x) = \sum_{i=0}^{r} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{(-1)^{r+1}k}{(r+1)!}(x-a)^{r+1}.$$

Define the parameter boundary

$$k_- := \|f\|_{C^r,M}(r+1)!(r+1)\big(1 + (a-a_0)^{-r-1}\big). \tag{A.29}$$

Note that if $y > 0$ then $\sum_{i=0}^{r} y^i \leqslant (r+1)\max\{1,\, y^{r+1}\} \leqslant (r+1)\big(1 + y^{r+1}\big)$. Setting $y := a - a_0$, we get from a direct estimate that

$$\begin{aligned} g_{k_-}(a_0) &\geqslant \frac{k_-}{(r+1)!}(a-a_0)^{r+1} - \sum_{i=0}^{r} \big|f^{(i)}(a)\big|(a-a_0)^i \\ &\geqslant \|f\|_{C^r,M}(r+1)\big(1 + (a-a_0)^{r+1}\big) - \|f\|_{C^r,M}\sum_{i=0}^{r}(a-a_0)^i \geqslant 0. \end{aligned}$$

By a symmetric lower bound, $g_{-k_-}(a_0) \leqslant 0$. Since $g_k(a_0)$ is continuous (indeed affine) with respect to k, the intermediate value theorem guarantees the existence of some $k_1 \in [-k_-, k_-]$ such that $g_{k_1}(a_0) = 0$. Next we estimate the C^r-norm of g_{k_1} on $[a_0, a]$. A straightforward computation yields, for each $0 \leqslant i \leqslant r$,

$$g_{k_1}^{(i)}(x) = \sum_{j=0}^{r-i} \frac{f^{(i+j)}(a)}{j!}(x-a)^j + \frac{(-1)^{r+1}k_1}{(r+1-i)!}(x-a)^{r+1-i}.$$

Since $|k_1| \leqslant k_-$ and k_- is proportional to $\|f\|_{C^r,M}$, with a proportionality factor depending only on r and $a - a_0$ (cf. (A.29)), each derivative $\big|g_{k_1}^{(i)}(x)\big|$ is bounded by a constant multiple of $\|f\|_{C^r,M}$. Maximizing these over $0 \leqslant i \leqslant r$ gives a uniform bound

$$\|g_{k_1}\|_{C^r,[a_0,a]} \leqslant C_-\|f\|_{C^r,M},$$

for some constant $C_- \geqslant 1$ depending only on r and $a - a_0$.

An entirely analogous argument on the right side yields an extension h_{k_2} on $[b, b_0]$ such that $h_{k_2}(b_0) = 0$ and $\|h_{k_2}\|_{C^r,[b,b_0]} \leqslant C_+\|f\|_{C^r,M}$ for some constant $C_+ \geqslant 1$ depending only on r and

$b_0 - b$. Finally, define the glued function F on $M_0 = [a_0, b_0]$ by

$$F(x) = \begin{cases} g_{k_1}(x) & \text{if } x \in [a_0, a), \\ f(x) & \text{if } x \in M, \\ h_{k_2}(x) & \text{if } x \in (b, b_0], \end{cases}$$

where the matching of the first r derivatives, at both points a and b, ensures that F is C^r on M_0. Note that $F(a_0) = g_{k_1}(a_0) = 0$ and $F(b_0) = h_{k_2}(b_0) = 0$, and moreover

$$\|F\|_{C^r, M_0} \leqslant \|g_{k_1}\|_{C^r, [a_0, a]} + \|f\|_{C^r, M} + \|h_{k_2}\|_{C^r, [b, b_0]} \leqslant D_r \|f\|_{C^r, M},$$

where $D_r := 1 + C_- + C_+$ depends only on r and the lengths of the intervals. This completes the proof. □

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